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Weighted Shapley Value for Multichoice Game

Chan-Chung Shen   Ying-Keh Wu*

Abstract

Multichoice game is constructed by Hsiao and Raghaven [1]. They use action vectors to replace sets to present the situation of cooperation of players in a game, and find the solution for multichoice game. Thus, we can extend a classical game into a multichoice game whose players have many kinds of work levels. Their weight system shows that each distinct player takes the same work level to mean the same behavior. But this cannot be applied in the games when different player has the variant job. Improved multichoice games offer a new weight system for multichoice games allowing each distinct player to have $m+1$ kinds of work levels without restricting the behavior of each work level. Meanwhile, we are also interested in the solution of improved multichoice games and we can reduce the solution found to Hsiao and Raghaven [1], Kalai and Samet's[2], and Shapley's [5] solutions as particular cases. Thus, the improved multichoice game has a wider sense of application.

1 Introduction

After the proposal of weighted Shapley value [2], Chih-Ru Hsiao and T.E.S. Raghaven [1] raise an new idea of game. They allowed each player of a game to take various kinds of action, and each kind of action presents the player's work level. However, these players coordinate a job in different work level, then by a characteristic function V the players get their payoffs. Question is: how much is the value of each kind of work level for distinct player of a game V?

In our daily life, we cannot avoid the chance to cooperate with others in the work that can't be done alone. For the purpose to complete a job, players in the game like to cooperate with somebody else. In Hsiao's paper [1], he gave an example about operating a harvest car, harvester. The harvester needs two people to operate. One person has to drive the car, and the other has to put away the crop. Suppose the player 1 is a girl that she is not strong enough to put away the crop, but she can drive. The other is a boy and he can not only drive but also put away the crop. In this example, we are able to use another boy to replace the girl they still can complete the job which is to harvest the crop.

*Department of Mathematics, Tunghai University, Taichung, 407, TAIWAN

However, here comes the new problem, if we do our own work and we cannot replace the position from each other. At the same time, we must cooperate to generate the profit. How should we share the profit fairly?

2 Definitions And Notations

In this section we introduce some terminologies needed for description of subquent sections.

Let $N = \{1, 2, \ldots, n\}$ be the grand coalition of n players. We allow each player to have $0, 1, 2, \ldots, m$ level of actions, where level 0 is doing nothing, while player j takes action k which means player j is doing some kind of work at level k. For notation convenience, we denote $x_j = k$. Let $\beta = \{0, 1, 2, \ldots, m\}$, and the action space of N is defined by $\beta^n = \{(x_1, x_2, \ldots, x_n) | x_i \in \beta, \forall i \in N\}$, and let $\beta^n \setminus \{\vec{0}\} = \beta^*$. Then we call that $\vec{x} = (x_1, x_2, \ldots, x_n)$ is a action vector of a game V. Now, given a coalition $S \subset N$, define $b(S) = (x_1, x_2, \ldots, x_n)$, where $x_j = \begin{cases} 1 & \text{if} \quad j \in S \\ 0 & \text{otherwise} \end{cases}$. Moreover, the *characteristic function* of the cooperative game is defined by

$$V : \beta^n \to \mathbb{R} \quad \text{such that } V(\vec{0}) = 0$$

Let $w_j(x_j) = \begin{cases} \lambda_{jk} & \text{if} \quad x_j = k \\ 0 & \text{if} \quad x_j = 0 \end{cases}$, for each $j \in N$, be the given weight of each player j, then it is reasonable to assume that $\lambda_{j0} \leq \lambda_{j1} \leq \cdots \leq \lambda_{jm}$, and represent them as a matrix below:

$$w(N) \equiv \begin{pmatrix} 0 & \lambda_{11} & \ldots & \lambda_{1m} \\ 0 & \lambda_{21} & \ldots & \lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \lambda_{n1} & \ldots & \lambda_{nm} \end{pmatrix} = M_{n \times (m+1)}$$

A game with such $w(N)$ is called an *improved multichoice game*. For this weight $w(N)$, for any $\vec{x} \in \beta^n$, we denote $||\vec{x}||_w = \sum_{r=1}^{n} w_r(x_r) = \sum_{\substack{r=1 \\ x_r = l}}^{n} \lambda_{rl}$, where $l \in \beta$, and $M_j(\vec{x}) = \{i | x_i \neq m, i \neq j\}$, for each $j \in N$.

Definition 2.1 *We say the action vectors $\vec{x}$ and $\vec{y}$ is* comparable *in the order relation "$\geq$", if $x_i \geq y_i$, x_i, $y_i \in \beta$ for every $i \in N$, means $\vec{x} \geq \vec{y}$.*

Clearly, the relation "$\geq$" satisfies *anti-symmetric* and *transitivity* which means it is a *partial order* relation.

Definition 2.2 *The* intersection *"$\cap$" of the action vectors $\vec{x}$ and $\vec{y}$, denoted by $\vec{z}$, is defined by*

$$\vec{x} \cap \vec{y} = \vec{z} = \{(z_1, z_2, \ldots, z_n) | z_i = \min\{x_i, y_i\}, \forall i \in N\}.$$

Definition 2.3 *An action vector $\vec{x}$ is called a* carrier *of V, if $V(\vec{x} \cap \vec{y}) = V(\vec{y})$, for all $\vec{y} \in \beta^*$. And we call $\vec{x^*}$ a* minimal carrier *of V, if $\vec{x^*} \geq \vec{x}$ implies $\vec{x^*} = \vec{x}$, where $\vec{x}$ is a carrier of V.*

Definition 2.4 *Player i is said to be a* dummy player *if $V((x_1, x_2, \ldots, x_i = k, \ldots, x_n)) = V((x_1, x_2, \ldots, x_i = 0, \ldots, x_n))$ for all $\vec{x} \in \beta^n$ and for all $k \in \beta$.*

3 Axioms and Main Results

In this section, we want to find out the solution for the improved situation. The sets of all *improved multichoice games*, denoted by G', can be identified by $G' \cong \mathbb{R}^{\beta^*}$. We consider the solution $\phi : G' \to M_{m \times n}$ as the following form:

$$\begin{aligned} \phi(V) &= \begin{pmatrix} \phi_{11}(V) & \ldots & \phi_{1n}(V) \\ \phi_{21}(V) & \ldots & \phi_{2n}(V) \\ \vdots & \ddots & \vdots \\ \phi_{m1}(V) & \ldots & \phi_{mn}(V) \end{pmatrix} \\ &= (\vec{\phi_1}(V), \vec{\phi_2}(V), \ldots, \vec{\phi_n}(V)) \end{aligned}$$

and

$$\vec{\phi_i}(V) = \begin{pmatrix} \phi_{1i}(V) \\ \phi_{2i}(V) \\ \vdots \\ \phi_{mi}(V) \end{pmatrix}$$

The value ϕ would like to satisfy some axioms analogous to the axioms of Shapley value. We list the axioms as follows:

Axiom 3.1 *Let $w(N)$ and $\vec{x}$ be given. If V is of the form*

$$V(\vec{y}) = \begin{cases} c > 0 & \text{if} \quad \vec{y} \geq \vec{x} \\ 0 & \text{if} \quad \vec{y} \ngeq \vec{x} \end{cases},$$

then $\phi_{x_j, j}(V)$ is proportional to $w_j(x_j)$.

Axiom 3.2 *If* $\vec{x^*}$ *is a carrier of* V *, then we have* $\sum_{x_j^* \in \vec{x^*}} \phi_{x_j^*,j}(V) = V(\vec{m})$.

That is the value ϕ is efficient.

Axiom 3.3 $\phi(V^1+V^2) = \phi(V^1)+\phi(V^2)$, *where* $(V^1+V^2)(\vec{x}) = V^1(\vec{x})+V^2(\vec{x})$.

That means solution ϕ is additive.

Axiom 3.4 *Given* $\vec{x^0} \in \beta^n$ *if* $V(\vec{x}) = 0$, *whenever* $\vec{x} \not\geq \vec{x^0}$, *then for each* $i \in N, \phi_{x_i,i}(V) = 0$, *for all* $x_i < x_i^0$.

That is, in the game V, it stipulates a minimal exertion from players, those who fail to meet this minimal level cannot be rewarded. The following lemma is easily proved.

Lemma 3.5 *There exists a unique minimal carrier of* V.

Remark 3.6 Let $H = \{\vec{h_k} | \vec{h_k} \in \beta^n$ is a carrier of V, $k = 1,2,\ldots,l\}$. Note that H is finite.

Choose $\vec{g} = \vec{h_1} \cap \vec{h_2} \cap \cdots \cap \vec{h_l} = \cap_{\vec{h_k} \in H} \vec{h_k}$. Then $\vec{g} = \{(g_1,g_2,\ldots,g_n) | g_i = \min\{h_{ki} | \forall k = 1,2,\ldots,l\}$, for all $i \in N\}$. Since $\vec{g}$ is the intersection of all carriers of V, $\vec{g}$ is also a carrier of V. Thus for each $\vec{h} \in H$, $\vec{g} \geq \vec{h}$ implies $\vec{g} = \vec{h}$. Now by the definition of minimal carrier, $\vec{g}$ is a minimal carrier of V.

By this lemma, we know each game $V \in G'$ has at least one carrier of V. Sometimes $\vec{m}$ can be selected as minimal carrier of a game V, if there is no other carrier in the game. A minimal carrier tells us the minimal level of total payoff of the project and overwork of players cannot generate more payoff then the total of the game.

For any $\vec{x} \neq \vec{0}, \vec{x} = (x_1,x_2,\ldots,x_n) \in \beta^n = \{0,1,2,\ldots,m\}^n$ we define

$$V_{\vec{x}}(\vec{y}) = \begin{cases} 1 & \text{if} \quad \vec{y} \geq \vec{x} \\ 0 & \text{if} \quad \vec{y} \not\geq \vec{x} \end{cases}$$

then the basis generating G' is given by the following theorem.

Theorem 3.7 *If for all* $j \in N$, $w_j(x_j) = \begin{cases} \lambda_{jk} & ,if \quad x_j = k \\ 0 & ,if \quad x_j = 0 \end{cases}$ *are given, then there exists a basis* $B = \{V_{\vec{x}} | \vec{x} \in \beta^n, \vec{x} \neq \vec{0}\}$ *generating* G'.

Theorem 3.8 *For any* $V \in G'$, *with* $V = \sum_{\vec{x} \in \beta^*} r_{\vec{x}} V_{\vec{x}}$, $\phi(V)$ *satisfies Axioms 3.1$\sim$ 3.4.*

Next, we like to prove the solution of improved multichoice game.

Theorem 3.9 *Suppose* $w_j(x_j) = \begin{cases} \lambda_{jk} & if \quad x_j = k \\ 0 & if \quad x_j = 0 \end{cases}$ *for all* $j \in N$ *are given, and* $\phi : G' \to M_{m\times n}$ *satisfies Axioms 3.1∼ 3.4. Then* $\phi(V)$ *is given by*

$$\begin{aligned}\phi_{ij}(V) &= \sum_{k=1}^{i} \sum_{\substack{x_j=k \\ \vec{x}\in\beta^*}} \left[\sum_{T\subseteq M_j(\vec{x})} (-1)^{|T|} \frac{w_j(x_j)}{\|\vec{x}\|_w + \sum_{r\in T} [w_r(x_r+1) - w_r(x_r)]} \right] \\ &\times \quad [V(\vec{x}) - V(\vec{x} - b(\{j\}))]\end{aligned} \tag{3.1}$$

The result establishes the solution for the improved situation.

4 Specification and Example

In this section we discuss the relations between multichoice games and classical games. In classical games, we know that each player has just two actions 0 and 1, no matter for the Shapley value or the weighted Shapley value. In Hsiao's paper, he has proved that the Shapley value is a special case of his multichoice game G, but the weighted Shapley value is not. In the weighted Shapley value each player takes the same action "1", but their weights are different from each other. But how is the improved multichoice game? The answer is positive that we can justify the weighted Shapley value is a special case of improved multichoice game G'. Next, we offer the proof.

Before giving the formal proof, we prove the following lemma.

Lemma 4.1

$$(\phi_w)_j(v) = \sum_{j\in S\subseteq N} \left[\sum_{T\subseteq N\setminus S} (-1)^{|T|} \frac{\lambda_j}{\sum_{i\in S\cup T} \lambda_i} \right] \times [v(S) - v(S \setminus \{j\})]$$

Suppose there is a game v, and weight system $w = (\lambda, \Sigma)$. Then the players in the game v can be seen as the players in the improved multichoice game V which has only two actions 0 and 1. Then we define the characteristic function of V as:

$$V((x_1, x_2, \ldots, x_n)) = v(S)$$

where for each $i \in N$, $x_i = 1$ if $i \in S$, and $x_i = 0$ otherwise. Then we can prove the following theorem.

Theorem 4.2 *If we define the weight function* $w_i(x_i) = \begin{cases} \lambda_i & if \quad x_i = 1 \\ 0 & if \quad x_i = 0 \end{cases}$ *, then* $\phi_{1,j}(V) = (\phi_w)_j(v)$, *for every* $i \in N$*, that is, we can express the weighted Shapley of* v *as the form of Shapley value for multichoice game* V*.*

Theorem 4.2 tells us that each classical game can be represented in the form of improved multichoice game which means $\Gamma \subset G'$.

Here, we present an example to figure out the specified property of the Improved Multichoice Games.

Example 4.3 A factory has three different production lines, a, b and c, and each of them produces different certain commodities. All of the production lines have three production levels. This factory makes revenue by the cooperative of the production lines. The following are their production levels.

a. The production lines a:

- 0: The line does not produce.
- 1: It produces rubber and bicycle tires.
- 2: It produces all kinds of tire of car.

b. The production lines b:

- 0: The line does not produce.
- 1: It produces the spare parts of bicycle or all kinds of spare parts of engine.
- 2: It produce all kinds of spare parts of car.

c. The production lines c:

- 0: The line does not produce.
- 1: It constitutes and produces bicycles.
- 2: It constitutes all kinds of cars.

And the revenue function is

$$V((x_a,x_b,x_c)) = 2x_a + x_b + 5x_c + 4\left\lfloor \frac{x_a x_c + x_b x_c}{2} \right\rfloor, \text{ Where } \lfloor\quad\rfloor \text{ is Gauss function.}$$

Now the owner wants to analyze each production levels. We can draw the table for the revenue.

$V((x_a,x_b,x_c))$		x_a		
x_b	x_c	0	1	2
0	0	0	2	4
	1	5	7	13
	2	10	16	22
1	0	1	3	5
	1	6	12	14
	2	15	21	27
2	0	2	4	6
	1	11	13	19
	2	20	26	32

The factory gives its weight:

$$w(N) \equiv \begin{pmatrix} 0 & \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{21} & \lambda_{22} \\ 0 & \lambda_{31} & \lambda_{31} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

Next, we compute $\phi(V)$:

$$\phi(V) \quad = \quad \begin{pmatrix} \phi_{11}(V) & \phi_{21}(V) & \phi_{13}(V) \\ \phi_{21}(V) & \phi_{22}(V) & \phi_{23}(V) \end{pmatrix} = \begin{pmatrix} 2\frac{1793}{2520} & 2\frac{62}{315} & 6\frac{425}{693} \\ 7\frac{841}{6930} & 6\frac{296}{385} & 17\frac{529}{1386} \end{pmatrix}$$

Finally, we check the efficiency of V:

$$\phi_{21}(V) + \phi_{22}(V) + \phi_{23}(V) = 7\frac{841}{6930} + 6\frac{296}{385} + 17\frac{529}{1386} = 32 = V(N)$$

This example demonstrates the cooperation among different kinds of work, and we apply the improved multichoice game to find the solution. Obviously, Hsiao's multichoice game cannot be applied in the Example 4, and we have posed another point of view to go with multichoice game.

5 Conclusion

In this thesis, we have established the value for improved multichoice game. It has the property that, except to be a dummy player, each player can decide which level of the game he wants to join. It, therefore, has a wider sense of application.

However, the objective is how to share the profit among these players of the coalition. Fortunately, we have found the approach to determine each player's value of the game.

By the results found, we are able to decide what kind or level of action will be more appropriate, or more effective. Our aim is that we like to find the optimal solution. By the way, we pose the game certain constraints, for instance, see Example, where the maximum revenue may not be the maximum profit for the game. Thus it seems a good direction to us to do further research in the future.

Appendix

A.1 Proof of Theorem 3.7

Proof: By Lemma 3.5, $\vec{x}$ is a carrier of $V_{\vec{x}}$, and by Axiom 3.1 we have

$$\phi_{x_j,j}(V_{\vec{x}}) = \frac{w_j(x_j)}{\sum\limits_{i\in N} w_i(x_i)} = \frac{\lambda_{jk}}{\sum\limits_{\substack{i\in N\\ x_i=l}} \lambda_{il}}, \text{ for each } j \in N,$$

where $k,l = 0,1,\ldots,m$. Furthermore, by Axiom 3.2, we have

$$\sum_{x_j\in\vec{x}} \phi_{x_j,j}(V_{\vec{x}}) = V_{\vec{x}}(\vec{m}) = 1$$

Let $\vec{x}_1 = (x_1,x_2,\ldots,x_j+1,x_{j+1},\ldots,x_n)$. Clearly, $\vec{x}_1$ is also a carrier of $V_{\vec{x}}$ and we have

$$\sum_{\substack{i\neq j\\ x_j\in\vec{x}}} \phi_{x_i,i}(V_{\vec{x}}) + \phi_{x_j+1,j}(V_{\vec{x}}) = V_{\vec{x}}(\vec{m}) = \sum_{x_i\in\vec{x}} \phi_{x_i,i}(V_{\vec{x}})$$

Hence $\phi_{x_j+1,j}(V_{\vec{x}}) = \phi_{x_j,j}(V_{\vec{x}})$. Similarly, $\phi_{l,j}(V_{\vec{x}}) = \phi_{x_j,j}(V_{\vec{x}})$, for all $l \geq x_j$, for each $j \in N$. By Axiom 3.4, $\phi_{l,j}(V_{\vec{x}}) = o$, for all $l < x_j$, for each $j \in N$. Thus we have

$$\vec{\phi}_j(V_{\vec{x}}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\lambda_{jk}}{\sum_{i\in N}\lambda_{il}} \\ \vdots \\ \frac{\lambda_{jk}}{\sum_{i\in N}\lambda_{il}} \end{pmatrix} \begin{matrix} \text{1st} \\ \\ \\ x_j\text{th} \\ \\ m\text{th} \end{matrix}$$

Denote

$$\vec{\phi}_j(r \cdot V_{\vec{x}}) = r\vec{\phi}_j(V_{\vec{x}}) = r \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\lambda_{jk}}{\sum_{i\in N}\lambda_{il}} \\ \vdots \\ \frac{\lambda_{jk}}{\sum_{i\in N}\lambda_{il}} \end{pmatrix}$$

As we know $G' \cong \mathbb{R}^{(m+1)^n-1}$, we need show that $B = \{V_{\vec{x}} | \vec{x} \in \beta^n, \vec{x} \neq \vec{0}\}$ is a basis for G'.

Assume there exists some $r_{\vec{x}} \neq 0$, and let for any $\vec{y} \in \beta^n$, $\sum_{\vec{x}\in\beta^*} r_{\vec{x}} V_{\vec{x}}(\vec{y}) \equiv 0$. Then we observe the following cases:

1. For $\vec{y} > \vec{x} \Rightarrow r_{\vec{x}} = 0$
2. For $\vec{y} < \vec{x} \Rightarrow V_{\vec{x}}(\vec{y}) = 0$
3. For $\vec{y} \not> \vec{x} \Rightarrow \vec{y} = \vec{x}$ or $\vec{y} \not\geq \vec{x}$.
   (a) If $\vec{y} \not\geq \vec{x}$, then $V_{\vec{x}}(\vec{y}) = 0$.
   (b) If $\vec{y} = \vec{x}$, then $\sum_{\vec{x}\in\beta^*} r_{\vec{x}} V_{\vec{x}}(\vec{y}) = r_{\vec{x}} V_{\vec{x}}(\vec{x}) = r_{\vec{x}} = 0$ This leads to a contradiction.

Hence for all $\vec{x} \in \beta^*$, $\{V_{\vec{x}}\}$ is linearly independent. Thus for every $V \in G'$, we can represent the game V as $V = \sum_{\vec{x}\in\beta^*} r_{\vec{x}} V_{\vec{x}}$. And, by Axiom 3.3,

$$\phi(V) = \phi(\sum_{\vec{x}\in\beta^*} r_{\vec{x}} V_{\vec{x}}) = \sum_{\vec{x}\in\beta^*} r_{\vec{x}} \phi(V_{\vec{x}})$$

□

A.2 Proof of Theorem 3.8

Proof: Let $\vec{x^*} \in \beta^*$ be the minimal carrier of V, for every $V \in G'$, define a linear transformation $h : G' \to G'$ by $h(V)(\vec{y}) = V(\vec{y} \cap \vec{x^*})$. Then for all $\vec{y} \in \beta^*$

$$h(V_{\vec{x}})(\vec{y}) = V_{\vec{x}}(\vec{y} \cap \vec{x^*}) = \begin{cases} 1 & ,\text{if} \quad \vec{y} \cap \vec{x^*} \geq \vec{x} \\ 0 & ,\text{if} \quad \vec{y} \cap \vec{x^*} \not\geq \vec{x} \end{cases}$$

Consider the following cases:

1. When $\vec{x^*} \geq \vec{x}$, we have $\vec{y} \cap \vec{x^*} \geq \vec{x}$ if and only if $\vec{y} \geq \vec{x}$.

2. When $\vec{x^*} \ngeq \vec{x}$, we have $\vec{y} \cap \vec{x^*} \ngeq \vec{x}$.

Thus $h(V_{\vec{x}}) = \begin{cases} V_{\vec{x}} & ,\text{if} \quad \vec{x^*} \geq \vec{x} \\ 0 & ,\text{if} \quad \vec{x^*} \ngeq \vec{x} \end{cases}$. Then for each $V_{\vec{x}} \in B$, we have

$$\sum_{x_j^* \in \vec{x^*}} \phi_{x_j^*,j}(V_{\vec{x}}) = \sum_{x_j \in \vec{x}} \phi_{x_j,j}(V_{\vec{x}}) = \sum_{x_j \in \vec{x}} \frac{w_j(x_j)}{\sum_{i \in N} w_i(x_i)} = V_{\vec{x}}(\vec{m}).$$

This satisfies Axiom 3.1. Moreover h is linear, and the range of h , $R(h)$, is generated by $B' = \{V_{\vec{x}} | \vec{x} \leq \vec{x^*}\}$

$$\begin{aligned} \sum_{x_j^* \in \vec{x^*}} \phi_{x_j^*,j}(V_{\vec{x}} + V_{\vec{y}}) &= \sum_{x_j^* \in \vec{x^*}} \phi_{x_j^*,j}(h(V_{\vec{x}} + V_{\vec{y}})) = h(V_{\vec{x}} + V_{\vec{y}})(\vec{m}) = h(V_{\vec{x}})(\vec{m}) + h(V_{\vec{y}})(\vec{m}) \\ &= \sum_{x_j^* \in \vec{x^*}} \phi_{x_j^*,j}(V_{\vec{x}}) + \sum_{x_j^* \in \vec{x^*}} \phi_{x_j^*,j}(V_{\vec{y}}) = \sum_{x_j^* \in \vec{x^*}} \left[\phi_{x_j^*,j}(V_{\vec{x}}) + \phi_{x_j^*,j}(V_{\vec{y}}) \right] \end{aligned}$$

This satisfies Axiom 3.3. For every given $V \in G', \vec{x^*}$ a is carrier of V.

$$\begin{aligned} \sum_{x_j^* \in \vec{x^*}} \phi_{x_j,j}(V) &= \sum_{x_j^* \in \vec{x}} \phi_{x_j,j}(\sum_{\vec{x} \in \beta^*} r_{\vec{x}} V_{\vec{x}}) = \sum_{x_j^* \in \vec{x^*}} \sum_{\vec{x} \in \beta^*} r_{\vec{x}} \phi_{x_j,j}(V_{\vec{x}}) \\ &= \sum_{\vec{x} \in \beta^*} r_{\vec{x}} \left(\sum_{x_j^* \in \vec{x^*}} \phi_{x_j,j}(V_{\vec{x}}) \right) = \sum_{\vec{x} \in \beta^*} r_{\vec{x}} V_{\vec{x}}(\vec{m}) = V(\vec{m}) \end{aligned}$$

This satisfies Axiom 3.2. Next, let $V(\vec{y}) = 0$, for any given $V \in G$, while $\vec{y} \ngeq \vec{x^*}$, $\vec{x^*}$ is a carrier of V. Let $V = \sum_{\vec{x} \in \beta^*} r_{\vec{x}} V_{\vec{x}}$, we consider the following conditions:

1. $\vec{x} \geq \vec{x^*} \Rightarrow \vec{y} \ngeq \vec{x} \Rightarrow V_{\vec{x}}(\vec{y}) = 0$

2. $\vec{x^*} > \vec{x}$ and $\vec{y} \geq \vec{x} \Rightarrow r_{\vec{x}} = 0$

3. $\vec{x} \ngeq \vec{x^*}$ and $\vec{x} \geq \vec{y} \Rightarrow V_{\vec{x}}(\vec{y}) = 0$

4. $\vec{x} \ngeq \vec{x^*}$ and $\vec{x} \ngeq \vec{y} \Rightarrow \vec{x} \ngeq \vec{y}$ and $\vec{y} \ngeq \vec{x}$, or $\vec{x} = \vec{y}$.

   (a) If $\vec{x} \ngeq \vec{y}$ and $\vec{y} \ngeq \vec{x}$, then $V_{\vec{x}} = 0$.

   (b) If $\vec{x} = \vec{y}$ then $V_{\vec{x}}(\vec{y}) = 1$.

Hence $V(\vec{y}) = \sum_{\vec{x} \in \beta^*} r_{\vec{x}} V_{\vec{x}}(\vec{y}) = r_{\vec{y}} V_{\vec{y}}(\vec{y}) = r_{\vec{y}} = 0$, for any $\vec{y} \ngeq \vec{x}$.
Then we conclude $\phi_{y_j,j}(V) = 0$, for all $j \in N$. This satisfies Axiom 3.4. □

A.3 Proof of Theorem 3.9

Proof: The value function of Theorem 3.8 was characterized in the proof as being the unique linear map $\phi : G' \to M_{m\times n}$ such that

$$\vec{\phi}_j(V_{\vec{x}}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\lambda_{jk}}{\sum_{i\in N}\lambda_{il}} \\ \vdots \\ \frac{\lambda_{jk}}{\sum_{i\in N}\lambda_{il}} \end{pmatrix} \begin{matrix} \text{1st} \\ \\ \\ \\ x_j\text{th} \\ \\ m\text{th} \end{matrix}, \text{ where } V_{\vec{x}}(\vec{y}) = \begin{cases} 1 & \text{,if } \quad \vec{y} \geq \vec{x} \\ 0 & \text{,if } \quad \vec{y} \ngeq \vec{x} \end{cases}. \tag{A.1}$$

The formula (3.1) is obviously linear in V. Since ϕ satisfies the Axioms 3.1$\sim$ 3.4 , to show (3.1) is sufficient to show (A.1). Given $\vec{x} \in \beta^*$ and $j \in N$, we need check

$$\begin{aligned} \phi_{x_j,j}(V_{\vec{x}}) &= \sum_{\substack{y_j = x_j \\ \vec{y}\in\beta^*}} \left[\sum_{T\subseteq M_j(\vec{x})} (-1)^{|T|} \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1) - w_r(x_r)]} \right] \\ &\times \quad [V_{\vec{x}}(\vec{y}) - V_{\vec{x}}(\vec{y} - b(\{j\}))] \end{aligned} \tag{A.2}$$

First, we consider for every $\vec{y} \in \beta^*$:

1. For all $\vec{y} \ngeq \vec{x} \Rightarrow V_{\vec{x}}(\vec{y}) = V_{\vec{x}}(\vec{y} - b(\{j\})) = 0$.
   Then for every $y_j, y_j - 1 < x_j$, we have $\phi_{y_j,j}(V_{\vec{x}}) = 0$, for all $y_j < x_j$.

2. For all $\vec{y} \geq \vec{x}$

   (a) When $V_{\vec{x}}(\vec{y}) - V_{\vec{x}}(\vec{y} - b(\{j\})) = 0$, then $V_{\vec{x}}(\vec{y}) = V_{\vec{x}}(\vec{y} - b(\{j\})) = 1 \Rightarrow y_j > x_j$

   (b) When $V_{\vec{x}}(\vec{y}) - V_{\vec{x}}(\vec{y} - b(\{j\})) = 1 \Rightarrow y_j = x_j$

Thus, when $V_{\vec{x}}(\vec{y}) - V_{\vec{x}}(\vec{y} - b(\{j\})) = 1$, it remains to show that the formula (A.2) reduce to

$$\phi_{x_j,j}(V_{\vec{x}}) = \frac{w_j(x_j)}{||\vec{x}||_w}, \text{ for each } j \in N \tag{A.3}$$

Let

$$C_j(\vec{y}) = \sum_{T\subseteq M_j(\vec{x})} \left[(-1)^{|T|} \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1) - w_r(x_r)]} \right] \tag{A.4}$$

then (A.2) can be represented as

$$\phi_{x_j,j}(V_{\vec{x}}) = \sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}}} C_j(\vec{y})\cdot 1 = \sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}}} C_j(\vec{y}) \tag{A.5}$$

Given any $j\in N$, assume $M_j(\vec{x}) = S$. By the inclusion-exclusion principle we have

$$\sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}}} C_j(\vec{y}) = \sum_{\substack{y_j=x_j\\ \vec{y}=\vec{x}}} C_j(\vec{y}) + \sum_{\substack{|T|=1\\ T\subseteq S}} (-1)^{|T|+1} \sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}+b(T)}} C_j(\vec{y}) + \cdots + \sum_{|T|=|S|} (-1)^{|T|+1} \sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}+b(T)}} C_j(\vec{y})$$

For the case $\vec{x}+b(T)=\vec{y}$, we have

$$\sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}+b(T)}} C_j(\vec{y}) = \frac{w_j(x_j)}{||\vec{x}+b(T)||_w} = \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1)-w_r(x_r)]} \tag{A.6}$$

However,

$$\begin{aligned}
\sum_{\substack{y_j=x_j\\ \vec{y}\geq\vec{x}}} C_j(\vec{y}) &= \sum_{\substack{y_j=x_j\\ \vec{y}=\vec{x}}} C_j(\vec{y}) + \sum_{\substack{T\subseteq S\\ T\neq\emptyset}} (-1)^{|T|+1} \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1)-w_r(x_r)]}\\
&= C_j(\vec{x}) + \sum_{\substack{T\subseteq M_j(\vec{x})\\ T\neq\emptyset}} (-1)^{|T|+1} \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1)-w_r(x_r)]}\\
&= \sum_{T\subseteq M_j(\vec{x})} \left[(-1)^{|T|} \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1)-w_r(x_r)]}\right]\\
&+ \sum_{\substack{T\subseteq M_j(\vec{x})\\ T\neq\emptyset}} \left[(-1)^{|T|+1} \frac{w_j(x_j)}{||\vec{x}||_w + \sum_{r\in T}[w_r(x_r+1)-w_r(x_r)]}\right]\\
&= \frac{w_j(x_j)}{||\vec{x}||_w} = \phi_{x_j,j}(V_{\vec{x}})
\end{aligned}$$

□

A.4 Proof of Lemma 4.1

Proof: Let $S,T \subseteq N$ and $|T| = t, |S| = s, |N| = n$.

$$\begin{aligned}
&\sum_{j\in S\subseteq N}\left[\sum_{T\subseteq N\setminus S}(-1)^t\frac{\lambda_j}{\sum\limits_{i\in S\cup T}\lambda_i}\right]\times[v(S)-V(S\setminus\{j\})] \\
=\ &\sum_{j\in S\cup T\subseteq N}\frac{\lambda_j}{\sum\limits_{i\in S\cup T}\lambda_i}\left[\sum_{H\subseteq S\cup T}(-1)^{s+t-h}v(H)\right]=\sum_{j\in R\subseteq N}\frac{\lambda_j}{\sum\limits_{i\in R}\lambda_i}\left[\sum_{H\subseteq R}(-1)^{r-h}v(H)\right] \\
=\ &\sum_{j\in R\subseteq N}(\phi_w)_j(u_R)\cdot\alpha_R=(\phi_w)_j(\sum_{R\subseteq N}\alpha_R u_R)=(\phi_w)_j(v)
\end{aligned}$$

□

A.5 Proof of Theorem 4.2

Proof: Since each player $j \in N$ just has only two actions , $x_j = 0$, 1. Define

$$V(\vec{x}) = v(S),\ \text{for all } \vec{x} = (x_1, x_2, \ldots, x_n) \in \beta^n = \{0,1\}^n,\ \text{where } S = \{i|x_i = 1, \forall i \in N\}.$$

Then

$$\begin{aligned}
\phi_{0,j}(V) &= 0,\ \text{clearly.} \\
\phi_{1,j}(V) &= \sum_{k=1}^{1}\sum_{\substack{x_j=k\\ \vec{x}\in\beta^n}}\left[\sum_{T\subseteq M_j(\vec{x})}(-1)^{|T|}\frac{w_j(x_j)}{||\vec{x}||_w+\sum\limits_{r\in T}[w_r(x_r+1)-w_r(x_r)]}\right]\times[V(\vec{x})-V(\vec{x}-b\{j\})]
\end{aligned}$$

Thus we represent the formula as following :

$$\begin{aligned}
\phi_{1,j}(V) &= \sum_{\substack{x_j=1\\ \vec{x}\in\beta^n}}\left[\sum_{T\subseteq M_j(x)}(-1)^{|T|}\frac{w_j(x_j)}{\|\vec{x}\|_w+\sum\limits_{r\in T}[w_r(x_r+1)-w_r(x_r)]}\right]\times[V(\vec{x})-V(\vec{x}-b\{j\})] \\
&= \sum_{\substack{j\in S\\ S\subseteq N}}\left[\sum_{\substack{j\notin T\\ T\subseteq N\setminus S}}(-1)^{|T|}\frac{\lambda_j}{\sum\limits_{i\in S\cup T}\lambda_i}\right][v(S)-v(S\setminus\{j\})]
\end{aligned}$$

Now we prove $\phi_{1,j}(V) = (\phi_w)_j(v)$ by induction. Let S, $T \subseteq N$,$s = |S|$, $t = |T|$, $n = |N|$.

(1) For $n = 1$, $N = \{1\}$. $\phi_{1,1}(V) = V(1) = v(\{1\}) = (\phi_w)_1(v)$

(2) For $n = 2$, $N = \{i, j\}$.

$$\begin{aligned}
\phi_{1,j}(V) &= \sum_{\substack{x_j=1 \\ \vec{x}\in\beta^n}} \left[\sum_{T\subseteq M_j(x)} (-1)^{|T|} \frac{w_j(x_j)}{\|\vec{x}\|_w + \sum\limits_{r\in T} [w_r(x_r+1) - w_r(x_r)]} \right] \times [V(\vec{x}) - V(\vec{x} - b\{j\})] \\
&= \sum_{\substack{j\in S \\ S\subseteq N}} \left[\sum_{\substack{j\notin T \\ T\subseteq N\setminus S}} (-1)^{|T|} \frac{\lambda_j}{\sum\limits_{i\in S}\lambda_i + \sum\limits_{i\in T}\lambda_i} \right] \times [v(S) - v(S\setminus\{j\})] \\
&= \sum_{\substack{j\in S \\ S\subseteq N}} \left[\sum_{\substack{j\notin T \\ T\subseteq N\setminus S}} (-1)^{|T|} \frac{\lambda_j}{\sum\limits_{i\in S\cup T}\lambda_i} \right] \times [v(S) - v(S\setminus\{j\})] \\
&= \left[(-1)^0 \frac{\lambda_j}{\lambda_j} + (-1)^1 \frac{\lambda_j}{\lambda_j + \lambda_i} \right] \times [v(j) - v(\emptyset)] + \left[\frac{\lambda_j}{\lambda_j + \lambda_i} \right] \times [v(\{i,j\}) - v(\{i\})] \\
&= [v(\{i,j\}) - v(\{j\}) - v(\{i\}) + v(\emptyset)] \frac{\lambda_j}{\lambda_j + \lambda_i} + [v(\{j\}) - v(\emptyset)] \\
&= \alpha_{\{i,j\}}(\phi_w)_j(u_{\{i,j\}}) + \alpha_{\{j\}}(\phi_w)_j(u_{\{j\}}) \\
&= (\phi_w)_j(\sum_{S\subseteq N} \alpha_S u_S) = (\phi_w)_j(v)
\end{aligned}$$

(3) For $n = k$, $N = \{1, 2, \ldots, k\} = K$. By Lemma 4.1

$$\begin{aligned}
\phi_{1,j}(V) &= \sum_{\substack{j\in S \\ S\subseteq K}} \left[\sum_{\substack{j\notin T \\ T\subseteq K\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S}\lambda_i + \sum\limits_{i\in T}\lambda_i} \right] \times [v(S) - v(S\setminus\{j\})] \\
&= \sum_{S\subseteq K} \alpha_S(\phi_w)_j(u_S) = (\phi_w)_j(\sum_{S\subseteq K}\alpha_S u_S) = (\phi_w)_j(v)
\end{aligned}$$

(4) For $n = k+1, N = \{1,2,\dots,k\} \cup \{k+1\} = K \cup \{k+1\}$.

$$
\begin{aligned}
\phi_{1,j}(V) &= \sum_{\substack{j\in S\\ S\subseteq K\{k+1\}}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\cup\{k+1\}\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T} \lambda_i} \right] \times [v(S) - v(S \setminus \{j\})] \\
&= \sum_{\substack{j\in S\\ S\subseteq K}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\cup\{k+1\}\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T} \lambda_i} \right] \times [v(S) - v(S \setminus \{j\})] \\
&+ \sum_{\substack{j\in S\\ S\cup\{k+1\}\subseteq K\cup\{k+1\}}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\cup\{k+1\}\setminus S\cup\{k+1\}}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}} \lambda_i} \right] \\
&\quad \times [v(S\cup\{k+1\}) - v(S\{k+1\} \setminus \{j\})]
\end{aligned}
$$

For the notation convenience we denote $S \cup \{k+1\} \subseteq K \cup \{k+1\} \equiv S \subseteq K$, where we still can tell out the form $\frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}} \lambda_i}$.

$$
\begin{aligned}
\phi_{1,j}(V) &= \sum_{\substack{j\in S\\ S\subseteq K}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T} \lambda_i} + \sum_{\substack{j\notin T\\ T\subseteq K\setminus S}} (-1)^{t+1} \frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}} \lambda_i} \right] \times [v(S) - v(S \setminus \{j\})] \\
&+ \sum_{\substack{j\in S\\ S\subseteq K}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}} \lambda_i} \right] \times [v(S\cup\{k+1\}) - v(S\cup\{k+1\} \setminus \{j\})] \\
&= \sum_{\substack{j\in S\\ S\subseteq K}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T} \lambda_i} [v(S) - v(S \setminus \{j\})] \right] \\
&+ \sum_{\substack{j\in T\\ S\subseteq K}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}} (-1)^{t+1} \frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}} \lambda_i} [v(S) - v(S \setminus \{j\})] \right] \\
&+ \sum_{\substack{j\in S\\ S\subseteq K}} \left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}} (-1)^t \frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}} \lambda_i} \times [v(S\cup\{k+1\}) - v(S\cup\{k+1\} \setminus \{j\})] \right]
\end{aligned}
$$

Hence we get the Shapley value for multichoice game:

$$\begin{aligned}\phi_{1,j}(V) &= \sum_{S\subseteq K}\alpha_S(\phi_w)_j(u_s) + \sum_{\substack{j\in S\\ S\cup\{k+1\}\subseteq K\cup\{k+1\}}}\left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}}(-1)^{t+1}\frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}}\lambda_i}\right][v(S)-v(S\setminus\{j\})]\\ &+ \sum_{\substack{j\in S\\ S\cup\{k+1\}\subseteq K\cup\{k+1\}}}\left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}}(-1)^{t}\frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}}\lambda_i}\right][v(S\cup\{k+1\})-v(S\cup\{k+1\}\setminus\{j\})]\end{aligned}$$

But here we still denote $S\cup\{k+1\}\subseteq K\cup\{k+1\}\equiv S\subseteq K$ for convenience.

$$\begin{aligned}\phi_{1,j}(V) &= \sum_{S\subseteq K}\alpha_S(\phi_w)_j(u_s)\\ &+ \sum_{\substack{j\in S\\ S\subseteq K}}\left[\sum_{\substack{j\notin T\\ T\subseteq K\setminus S}}\left[(-1)^t v(S\cup\{k+1\})+(-1)^{t+1}v(S\cup\{k+1\}\setminus\{j\})\right.\right.\\ &+ \left.\left.(-1)^{t+1}v(S)+(-1)^{t+2}v(S\setminus\{j\})\right]\frac{\lambda_j}{\sum\limits_{i\in S\cup T\cup\{k+1\}}\lambda_i}\right]\end{aligned} \tag{A.7}$$

By Lemma 4.1 we can write (A.7) as follow:

$$\begin{aligned}&= \sum_{S\subseteq K}\alpha_S(\phi_w)_j(u_s) + \sum_{S\cup\{k+1\}\subseteq K\cup\{k+1\}}\alpha_{S\cup\{k+1\}}(\phi_w)_j(u_{S\cup\{k+1\}})\\ &= \sum_{S\subseteq K\cup\{k+1\}}\alpha_S(\phi_w)_j(u_S)\end{aligned}$$

□

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多重選擇局的夏普萊值

沈正中　　吳英格*

摘　　要

賽局理論有許多方面的應用，本文所討論的多重選擇局的加權夏普萊值，則專為討論合作局的情況。

當一群玩家參與一個賽局時，若其中有由二人以上的結盟出現，常會帶來超出該盟中成員原本所擁有的利益。但在帶來額外利益的同時，該同盟應如何去合理與適當地分配這些利益，從而決定盟隊成員在賽局中的價值，則是本文的論點所在。

古典的賽局理論中，合作關係僅止於玩家參不參與盟隊，並未給予玩家參與程度上的區別。多重選擇賽局則給予玩家有選擇參與的程度。但在這同時，它也同樣涉及如何決定這些玩家所採取的參與程度在賽局中的價值問題。本文最主要的理論即是提供其分配的方法，來決定每一個玩家的每一種參與程度在賽局中的價值。並由於方法的改良更使得 Hsiao[1]、Kalai[2] 與 Shapley[5]的方法，成為本法中的特殊情況。

* 東海大學數學系