The Covariant Entropy Bound and the Higher-Dimensional FRW Universe

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Abstract

We calculate the covariant entropy bound for the Friedmann-Robertson-Walker(FRW) universe in higher-dimensional space-time. Let $a(t)$ and $b(t)$ represent the scale factor of the four-dimensional space-time and the extra dimensions respectively. Assuming $b(t) = \kappa a^n(t)(n < 0)$, then some constraints are obtained.

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1 Introduction

Whenever a system suffer a gravitational collapse, the total entropy must be defined as the sum of the matter-entropy S_m plus the entropy of the black hole S_{BH}, i.e. $S = S_m + S_{BH}$. Hawking [1] showed that the black hole entropy is given by

$$S_{BH} = \frac{A}{4G} \tag{1}$$

where A is the area of the event horizon of the black hole. For any weakly gravitating matter system in asymptotically flat space in four-dimensional space-time, Bekenstein (1981) [2] proposed the entropy bound

$$S_m \leq 2\pi ER, \tag{2}$$

where E is the total mass-energy of the matter system and R is the radius of the smallest sphere that fits around the matter system. Note that the bound from above in the inequality (2) does not contain the Newton's constant G at all. This entropy bound is important because it is an attempt to set limits on the entropy of the system that is characterized by physical parameters such as energy and the size of the system.

Another kind of the entropy bound was proposed by Susskind in 1995 [3]. The maximum entropy of a system that can be enclosed by a spherical surface of the area A is given by

$$S \leq \frac{A}{4G}. \tag{3}$$

This is known as the spherical entropy bound. It can also be extended to a much more general bound called the space-like entropy bound.[4] For a compact portion of equal time spatial hypersurface in space-time with volume V and boundary B of area $A(B)$, then the total entropy inside the boundary is bounded by

$$S(V) \leq \frac{A(B)}{4G}. \tag{4}$$

Though this bound works in many systems, counterexamples can be found where it does not apply, for example, in cosmological scenarios or for strongly gravitational system[4]. The failure of the space-like entropy bound lead Bousso to propose a more suitable generalization which is known as the covariant entropy bound.[5]

On the other hand, there is a principle that change our thinking radically about the counting of degrees of freedom of physical system. How many degrees of freedom are there in which the entropy or information content is stored at the most fundamental levels? 't Hooft (1993)[6] and Susskind (1995)[3] presented the so-called holographic principle which answers this question in terms of the area of surfaces in space-time. The holographic principle could be possibly formulated as follows:

The full physical description of some given region of volume V, in an D-dimensional universe, with $(D-1)$-dimensional boundary $B \equiv \partial V$, can be reflected in processes taken place in B.

From a fundamental point of view, the entropy bounds are likely to be a more or less straightforward consequence of this principle. Despite in the absence of a well formulated holographic principle, entropy bounds can be extremely useful as heuristic tools for the task of clarifying the apparent contradictions between Quantum Mechanics and General Relativity.

In order to solve the difficulties encountered in the space-like entropy bound, Bousso proposed the covariant entropy bound which can be stated as follows:

Let A be the area of a connected $(D-2)$-dimensional spatial surface B contained in the D-dimensional space-time M. A $(D-1)$-dimensional hypersurface L is called a light-sheet of B if it is generated by surface orthogonal null geodesics with nonpositive expansion. Then the total entropy S, contained on L, satisfy the inequality

$$S(L) \leq \frac{A(B)}{4G}. \tag{5}$$

Furthermore, there is also a stronger version of the covariant entropy bound which is proposed in [7] known as the generalized covariant entropy bound. In this version the light rays in L are allowed to stop before they reach the caustic and in this way they define a new surface B' of the area $A(B')$. Then we have

$$S(L) \leq \frac{A(B) - A(B')}{4G}. \tag{6}$$

Eq.(6) reduces to Eq.(5) for the special case that $A(B') = 0$.

However, A. Masoumi and S.D. Mathur [8] proposed a possible violation of the covariant entropy bound. They suggested that the entropy density at high energy density ρ should be given by the expression $s = K\sqrt{\rho/G}$. On the other hand the covariant entropy bound requires that the entropy on a light sheet be bounded by $A/4G$, where A is the area of the boundary of the sheet. They found that a suitably chosen cosmological geometry, the above expression for s violates the covariant entropy bound.

In this paper we calculate the covariant entropy bound for the higher-dimensional spatially flat Friedmann-Robertson-Walker(FRW) universe and find that some constraints should exist.

2 Higher-Dimensional FRW Universe

The metric of the spatially flat Friedmann-Robertson-Walker universe in (4+m)-dimensional space-time can be expressed as

$$ds^2 = -dt^2 + a^2(t)[\,dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)] + b^2(t)\sum_{j=1}^{m} dy_j^2 \tag{7}$$

where $a(t)$ and $b(t)$ represent the scale factors of the four-dimensional space-time and the extra dimensions respectively. Eq.(7) can be rewritten as

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} dx_i^2 + b^2(t)\sum_{j=1}^{m} dy_j^2, \tag{8}$$

so that the directions t, x_i, y_j are all orthogonal to each other. The light sheet we use will be confined to the interval

$$t_0 - \Delta t \le t \le t_0 \tag{9}$$

where we will take the limit of small Δt. The metric is subject to Einstein's equations, which are second order equations for the metric components. We can choose the $a(t)$, $b(t)$, $\dot{a}(t)$, $\dot{b}(t)$ at time t_0 as:

$$a(t_0) \equiv a_0, \quad b(t_0) \equiv b_0, \quad \frac{\dot{a}(t_0)}{a(t_0)} \equiv \gamma_0, \quad \frac{\dot{b}(t_0)}{b(t_0)} \equiv \beta_0 \tag{10}$$

which subject only to the constraint set by the Einstein equation $G_t^t = 8\pi G T_t^t$:

$$-\frac{1}{2}\left(3\frac{\dot{a}}{a} + m\frac{\dot{b}}{b}\right)^2 + \frac{1}{2}\left(3\frac{\dot{a}^2}{a^2} + m\frac{\dot{b}^2}{b^2}\right) = -8\pi G\rho. \tag{11}$$

This constraint gives

$$-\frac{1}{2}\left(3\gamma_0 + m\beta_0\right)^2 + \frac{1}{2}\left(3\gamma_0^2 + m\beta_0^2\right) = -8\pi G\rho_0. \tag{12}$$

where $\rho_0 = \rho(t_0)$. The G_k^k (there is no sum over k) equations for the space directions we have

$$\begin{aligned} & \frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\left(3\frac{\dot{a}}{a} + m\frac{\dot{b}}{b}\right) - \frac{\dot{a}^2}{a^2} \\ & -\frac{1}{2}\left[2\left(3\frac{\ddot{a}}{a} + m\frac{\ddot{b}}{b}\right) + \left(3\frac{\dot{a}}{a} + m\frac{\dot{b}}{b}\right)^2 - \left(3\frac{\dot{a}^2}{a^2} + m\frac{\dot{b}^2}{b^2}\right)\right] \\ = \; & -8\pi G T_k^k \quad (k = 1, 2, 3), \end{aligned} \tag{13}$$

and

$$\begin{aligned} & \frac{\ddot{b}}{b} + \frac{\dot{b}}{b}\left(3\frac{\dot{a}}{a} + m\frac{\dot{b}}{b}\right) - \frac{\dot{b}^2}{b^2} \\ & -\frac{1}{2}\left[2\left(3\frac{\ddot{a}}{a} + m\frac{\ddot{b}}{b}\right) + \left(3\frac{\dot{a}}{a} + m\frac{\dot{b}}{b}\right)^2 - \left(3\frac{\dot{a}^2}{a^2} + m\frac{\dot{b}^2}{b^2}\right)\right] \\ = \; & -8\pi G T_k^k \quad (k = 4, 5 ... 3 + m). \end{aligned} \tag{14}$$

Assuming that the evolution of the full cosmology is a perfect fluid with $p = \rho$, we can solve Eq.(13) and (14),and yields finite values of the $\ddot{a}$,$\ddot{b}$. Thus in the interval given by Eq.(9) we get

$$a(t) \approx a_0 \left[1 + \gamma_0 \left(t - t_0\right)\right] \tag{15}$$

$$b(t) \approx b_0 \left[1 + \beta_0 \left(t - t_0\right)\right] \tag{16}$$

in which we will take

$$\gamma_0 > 0, \tag{17}$$

so that the slice $t_0 - \Delta t \leq t \leq t_0$ represents a segment of an expanding cosmology.

3 The Covariant Entropy Bound

Now we are in the (4+m)-dimensional space-time and consider a (2+m)-dimensional space-like hypersurface B. This hypersurface may be closed (i.e.without boundary) or it may be open (i.e. with boundary). Let A be the area of B. Let the B be a cuboid in the direction $x^2, x^3, y^1...y^m$, spanning the coordinate ranges

$$0 \leq x^i \leq L_i, \quad 0 \leq y^j \leq L'_j. \tag{18}$$

All points on this cuboid are at a fixed value of time t and space coordinate x^1, i.e.

$$t = t_0, \quad x^1 = x^1_0. \tag{19}$$

Our metric now is given by Eq.(8). The area of this hypersurface B is

$$A = \prod_{i=2}^{3} \prod_{j=1}^{m} L_i L'_j a(t_0) b(t_0) = \prod_{i=2}^{3} \prod_{j=1}^{m} L_i L'_j a_0 b_0. \tag{20}$$

At each point of the hypersurface B we look for a null geodesic leaving the hypersurface, in a direction that is orthogonal to the hypersurface. Each geodesic remains at a fixed value of the coordinates $x^2, x^3, y^1...y^m$. The change of x^1 is found by requiring $ds = 0$ in the metric given by Eq.(8), i.e.

$$\frac{dx^1}{dt} = \frac{1}{a(t)}. \tag{21}$$

We require that the set of null geodesics constructed in this way will be nondiverging as we move away from the hypersurface B. In other words, suppose we consider a small area element dA on B. A null geodesic will start from a point $(x^2_0, x^3_0, y^1_0...y^m_0)$ on B. We follow these geodesics for an affine distance λ, the transverse area spanned by the geodesics will have a value $dA(\lambda)$ and require that

$$\frac{dA(\lambda)}{d\lambda} \leq 0. \tag{22}$$

The geodesic starts with a tangent vector which has with nonzero components

$$\frac{dx^1}{d\lambda} < 0, \quad \frac{dt}{d\lambda} < 0, \tag{23}$$

so it heads to the past, in the direction of decreasing x^1. The area $dA(t)$ decreasing along the geodesics heading to the past is equivalent to

$$\frac{dA}{dt} = \frac{d}{dt}\left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a(t_0) b(t_0)\right) = \left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0\right)(2\gamma_0 + m\beta_0) > 0 \tag{24}$$

which is a result deriving from Eq.(15) and (16).

We follow the null geodesics described above up to the point where they reach a caustic; i.e.the point where the separation of neighboring geodesics goes to zero. The surface spanned by the null rays emanating from B, followed up to any point before meeting a caustic, defines a light sheet. We now consider the entropy S_{sheet} on this light sheet. This can also be defined by the entropy that crosses the light sheet from one side to the other.

We compute the entropy passing through the light sheet as follows. On the spatial slice at time t, consider the slice of space given by

$$x_0^1 - \Delta x^1 \leq x^1 \leq x_0^1, \tag{25}$$

$$0 \leq x^i \leq L_i \ (i = 2, 3), \qquad 0 \leq y^j \leq L'_j \ (j = 1, 2...m) \tag{26}$$

where

$$\Delta x^1 = \frac{\Delta t}{a_0} \tag{27}$$

The entropy passing through our light sheet is equal to the entropy present on this slice of space. The proper volume of this slice is

$$\Delta V = a_0 \Delta x^1 \left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0\right) = \Delta t \left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0\right). \tag{28}$$

The entropy density on the slice is

$$s = K\sqrt{\frac{\rho_0}{G}} \tag{29}$$

where we have used the same entropy density in [8] is used. K is a constant of order of unity. Thus the entropy passing through our light sheet is

$$S_{sheet} = s\Delta V = K\sqrt{\frac{\rho_0}{G}}\Delta t \left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0\right). \tag{30}$$

The generalized version of the covariant entropy bound is proposed in [7]. We will use the form

$$S_{sheet} \leq S_{bound} = \frac{A - A'}{4G}. \tag{31}$$

where $A' < A$ because the light rays are converge along their paths. The area A is given by Eq.(20) while the area A' is the area of the surface at the lower end of the light sheet. We have

$$\begin{aligned} A' &= \prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a(t_0 - \Delta t) b(t_0 - \Delta t) \\ &\approx \prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0 (1 - \gamma_0 \Delta t)(1 - \beta_0 \Delta t) \\ &\approx \left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0\right)(1 - \Delta t(2\gamma_0 + m\beta_0)) \end{aligned} \tag{32}$$

Then we get

$$S_{bound} = \frac{A - A'}{4G} = \frac{\left(\prod_{i=2}^{3}\prod_{j=1}^{m} L_i L'_j a_0 b_0\right)\Delta t(2\gamma_0 + m\beta_0)}{4G}. \tag{33}$$

Thus we can find that

$$\frac{S_{sheet}}{S_{bound}} \equiv r = \frac{4K\sqrt{\rho_0 G}}{(2\gamma_0 + m\beta_0)}. \tag{34}$$

Substituting ρ_0 from Eq.(12), we get

$$r = \frac{K}{\sqrt{\pi}} \frac{\left[(3\gamma_0 + m\beta_0)^2 - (3\gamma_0^2 + m\beta_0^2)\right]^{\frac{1}{2}}}{(2\gamma_0 + m\beta_0)} \tag{35}$$

Similar to G. Sarma [9], we assume a relation between two scale factors $a(t)$ and $b(t)$,

$$b(t) = \kappa a^n(t), \qquad n < 0, \tag{36}$$

where κ and n are constant. In the work of G. Sarma [9], he considered the spatially flat and isotropic universe in five-dimensional FRW metric, assumed the relation $b(t) = \kappa a^n(t)$, and claimed that n must be less than zero and have a condition $0 > n > -3$ for accelerating universe. Substituting the relation given in Eq.(36) into (10),we will have

$$\beta_0 = n\gamma_0, \tag{37}$$

and then

$$r = \frac{K}{\sqrt{\pi}} \frac{[6 + 6mn + (m^2 - m)n^2]^{\frac{1}{2}}}{(2 + mn)}, \tag{38}$$

where the relation given in Eq.(17), $\gamma_0 > 0$, is used. If the generalized covariant entropy bound holds true, r should be less than one, i.e.

$$1 \geq r, \tag{39}$$

Then it follows that

$$0 > n > -\frac{2}{m}, \tag{40}$$

$$6 + 6mn + (m^2 - m)n^2 \geq 0. \tag{41}$$

In the five-dimensional case, $m = 1$, we get the constraints

$$0 > n > -1 \tag{42}$$

and

$$K < \sqrt{\pi}(2 + n)/(6 + 6n)^{\frac{1}{2}}. \tag{43}$$

4 Concluding Remarks

We have calculated the generalized covariant entropy bound for higher-dimensional FRW universe and obtained three constraints listed in Eq. (39) to (41).This is consistent with the work of G. Sarma [9]. We do not claim that there are some violations of the covariant entropy bound in higher-dimensional FRW universe. However, if the covariant entropy bound holds true, nature may choose the right one among all possibilities. The covariant entropy bound may give us some informations in the higher-dimensional space-time.

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協變熵限和高維 **FRW** 宇宙

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摘 要

我們計算高維時空 FRW 宇宙的協變熵限。令 a(t)和 b(t)分別代表為四維時空和額外維度的刻度因子，若$b(t) = \kappa a^n(t)$，則我們會找到一些限制條件。

關鍵字: 協變熵限、 高維 **FRW** 宇宙