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A New Approach to Classical Multiable-variable Calculus

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Abstract

How to define the "fraction" $\frac{f(x,y)-f(a,b)}{<x-a,y-b>}$, where $< x-a, y-b >$ represents an arbitrary vector? In this manuscript we try to exploit it directly as the definition of derivatives for two variable functions. Based on this concept we obtain various forms of the Mean Value(Vector) Theorems and the Fundamental Theorems of Calculus for multiple integrals. We hope they are able to be helpful for those people in learning and teaching multi-variable calculus.

1 Introduction

How to define the vector division has been plagued by people. In this paper, we give a new definition of the fraction $\frac{f(x,y)-f(a,b)}{<x-a,y-b>}$, where $< x-a, y-b >$ represents an arbitrary vector, and use it to discuss the differentiability and integrability of two or more variables function. Using the definition, our result is better than Theorem 12.12 of [1] (pp. 357), and it is easier to explain the definition 3 in [2] (pp. 455). In section 2, we define the line derivector, plane derivector and use it to explain the chain rule for differentiation of multi-variables function, differential and the mean vector theorem. In section 3, we define the line integral, plane integral, surface integral and solid integral on the oriented plane cure, oriented plane region and oriented space box, respectively. Finally, we discuss various fundamental theorems of calculus for multiple integrals.

2 Differentiations

2.1 Derivatives and derivectors

We define various derivatives for multi-variable functions—Line derivectors, Line derivatives, Plane derivatives, Surface derivectors, Surface derivatives and Solid derivatives.

i) The line derivector for a 2-variable function $f(x, y)$ at (a, b)

Firstly,let us recall that the differentiation of single variable function $f(x)$ at a. Consider the equality

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a), x \neq a$$

we say that $f(x)$ has derivative at a if the fraction $\frac{f(x)-f(a)}{x-a}$ is defined and has limit at a.

For the differentiation of two-variable function $f(x, y)$ at (a, b), it seems that we can analogously consider the equality

$$f(x, y) - f(a, b) = \frac{f(x, y) - f(a, b)}{< x - a, y - b >} \cdot < x - a, y - b >, x \neq a, y \neq b$$

can we say that $f(x, y)$ has "derivative" at (a, b),if the new "fraction" $\frac{f(x,y)-f(a,b)}{<x-,y-b>}$ is defined appropriately and has double limit at (a, b)? Thus how to define the new fraction $\frac{f(x,y)-f(a,b)}{<x-a,y-b>}$? It will be the main aim of this paper.

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There are lot of possibilities to define the new fraction, but there is only one that will bring us plenty of theorems (e.g. see Theorem 2.1, 2.3, 2.6, MVT's, FTC's etc., below) about calculus of multi-variable functions. We define $\frac{f(x,y)-f(a,b)}{<x-a,y-b>}$ as follow

$$\frac{f(x,y)-f(a,b)}{<x-a,y-b>} \triangleq < \frac{f(x,y)-f(a,y)}{x-a}, \frac{f(a,y)-f(a,b)}{y-b} >, \text{ when } x \neq a, y \neq b$$

Is the definition well-defined? Yes, since the equality

$$f(x,y)-f(a,b) = < \frac{f(x,y)-f(a,y)}{x-a}, \frac{f(a,y)-f(a,b)}{y-b} > \cdot < x-a, y-b >$$

holds. Now with the definition of new fraction we can say that the given function $f(x,y)$ has the derivative at (a,b) if we assume further that the new fraction has double limit at (a,b).

Definition 2.1 (Line Derivector of $f(x,y)$ at (a,b)). *Let $f(x,y)$ be a 2-variable function defined on a plane convex open region D containing a point (a,b). We define* the line derivector for $f(x,y)$ at (a,b) *by the double limit*

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{<x-a,y-b>}$$

where $(x,y) \in D, x \neq a, y \neq b$, provided that the limit exists. To replace the vector $<dx,dy>$ in the denominator means that it concerns not only the changes in the two independent variables x,y but also those in their directions. We will denote the limit by $\left.\frac{df(x,y)}{<dx,dy>}\right|_{(a,b)}$.

Remark 2.1. For the case $\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{<y-b,x-a>}$, we can similarly define and get similar results.

$$\frac{f(x,y)-f(a,b)}{<y-b,x-a>} \triangleq < \frac{f(x,y)-f(x,b)}{y-b}, \frac{f(x,b)-f(a,b)}{x-a} >$$

Theorem 2.1. *Let $f(x,y)$ be a function defined on a planar convex open region D, containing (a,b), having partial derivatives at (a,b), and its partial derivatives $f_x(a,y), f_y(x,b)$ are continuous at (a,b). Then $f(x,y)$ has line derivector at(a,b), and the line derivector $\left.\frac{df(x,y)}{<dx,dy>}\right|_{(a,b)}$ is equal to $< f_x(a,b), f_y(a,b) >$.*

Proof. The existence of the line derivector can be proved through the proof the equality $\left.\frac{df(x,y)}{<dx,dy>}\right|_{(a,b)}$ $=< f_x(a,b), f_y(a,b) >$. By the definition and $f_x(a,y)$, $f_y(x,b)$ are continuous at (a,b), we have

$$\begin{aligned} & \lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{<x-a,y-b>} \\ = & \lim_{<x,y>\to<a,b>} < \frac{f(x,y)-f(a,y)}{(x-a)}, \frac{f(a,y)-f(a,b)}{(y-b)} > \\ = & < \lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,y)}{(x-a)}, \lim_{<x,y>\to<a,b>} \frac{f(a,y)-f(a,b)}{(y-b)} > \\ = & < f_x(a,\breve{b}), f_y(a,b) > = < f_x(a,b), f_y(a,b) > \end{aligned}$$

Similarly, we have

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{<y-b,x-a>} = < f_y(a,b), f_x(a,b) >$$

□

Remark 2.2. The line derivector of $f(x,y)$ at (a,b) turns out the well known gradient vector of $f(x,y)$ at (a,b), which is also denoted by $\nabla f(a,b)$.

Remark 2.3. Note that $< f_x(a,b), f_y(a,b) >$ is not the gradient vector of the function $f(x,y)$ at (a,b) if it doesn't get from above procedure. For instance, consider the following function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

Then the vector $< f_x(0,0), f_y(0,0) >=< 0,0 >$ is not the gradient vector of the considered function at $(0,0)$, since the function doesn't have line derivector at(a,b).

Corollary 2.2. *If $f(x,y,z)$ has line derivector at(a,b,c), then*

$$\left.\frac{df(x,y,z)}{< dx, dy, dz >}\right|_{(a,b,c)} =< f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) > .$$

Remark 2.4. Notice that, analogously to the case of two variable function, from the equality

$$\begin{aligned} & f(x,y,z) - f(a,b,c) \\ = \ & < \frac{f(x,y,z)-f(a,y,z)}{x-a}, \frac{f(a,y,z)-f(a,b,z)}{y-b}, \frac{f(a,b,z)-f(a,b,c)}{z-c} > \cdot < x-a, y-b, z-c > \end{aligned}$$

We define

$$\frac{f(x,y,z)-f(a,b,c)}{< x-a, y-b, z-c >} \triangleq < \frac{f(x,y,z)-f(a,y,z)}{x-a}, \frac{f(a,y,z)-f(a,b,z)}{y-b}, \frac{f(a,b,z)-f(a,b,c)}{z-c} >$$

Geometric Interpretation of the "fraction" $\frac{f(x,y)-f(a,b)}{<x-a,y-b>}$

$$\frac{f(x,y)-f(a,b)}{< x-a, y-b >} = \frac{BC}{\overrightarrow{AC}} = \frac{BD+CD}{\overrightarrow{AC}} =< \frac{BD}{x-a}, \frac{CD}{y-b} >$$

where $x-a = FD \perp BC$.

Geometric interpretation of line derivector

The Line derivector of $z = f(x,y)$ at (a,b) is a vector indicating the "slope vector" of the tangent plane to the surface $z = f(x,y)$ at (a,b). It just equals to take both partial derivatives of $f(x,y)$ at (a,b) for each component of the plane vector, the difference only in that its first component is taken by the double limit.

Recall that the equation of the tangent plane for surface $z = f(x,y)$ at (a,b) is $z - f(a,b) = < f_x(a,b), f_y(a,b) > \cdot < x-a, y-b >$. Notice that a plane needs two intersecting straight lines to be determined uniquely.

Physical interpretation of line derivector

From the equality

$$f(x,y) - f(a,b) = \frac{f(x,y)-f(a,b)}{< x-a, y-b >} \cdot < x-a, y-b >, x \neq a, y \neq b$$

we can interpret the new fraction $\frac{f(x,y)-f(a,b)}{<x-a,y-b>}$ runs as the force done the work $f(x,y)-f(a,b)$ due to the displacement $< x-a, y-b >$.

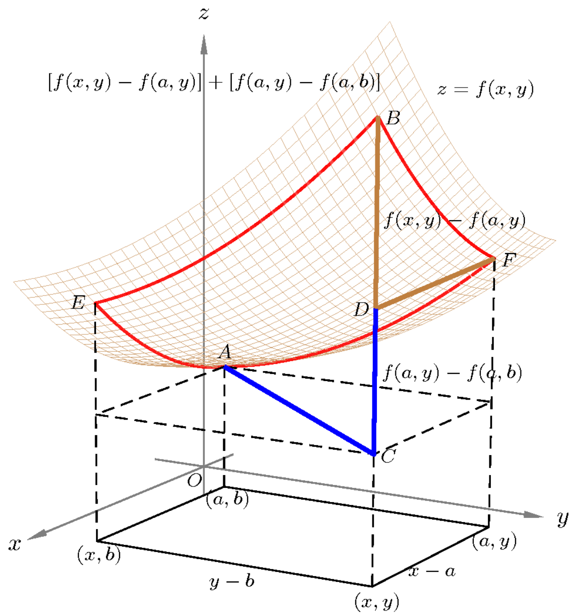

Figure 2.1: Geometric Interpretation of the fraction

Definition 2.2 (Line Differentiability)**.** *We say that $f(x, y)$ is line differentiable at (a, b) if the function $f(x, y)$ has line derivector at (a, b).*

Theorem 2.3. *The function $f(x, y)$ is line differentiable at (a, b) if and only if both its partial derivatives are partially continuous at (a, b).*

Proof. As easily to know that $\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,y)}{(x-a)}$ exists is equivalent to that $f_x(a, y)$ is continuous at (a, b). □

Remark 2.5.

1. The definition of line differentiability of $f(x, y)$ at (a, b) is simple, similar to that of single-variable function $f(x)$ at a, $f(x)$ is differentiable at a if it has derivative at a.

2. If $\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,y)}{x-a}$ exists, then $\lim_{x\to a} \frac{f(x,b)-f(a,b)}{x-a}$ exists, but not conversely. (For "not conversely", see Example 2.3 in the following.)

Corollary 2.4. *The function $f(x, y, z)$ is line differentiable at (a, b, c) if and only if it has line derivector at (a, b, c),; ie, the following limits*

$$\lim_{<x,y,z>\to<a,b,c>} \frac{f(x, y, z) - f(a, y, z)}{x - a}, \quad \lim_{<x,y,z>\to<a,b,c>} \frac{f(x, y, z) - f(x, b, z)}{y - b}$$

and

$$\lim_{<x,y,z>\to<a,b,c>} \frac{f(x, y, z) - f(x, y, c)}{z - c}$$

exists. That is, the function $f(x, y, z)$ is line differentiable at (a, b, c) if and only if its partial derivatives $f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)$ are all partially continuous at (a, b, c).

ii) Line derivatives for 2-variable functions

We define the line derivative for $f(x,y)$ at (a,b) by the double limit

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{ds} \triangleq \lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{\sqrt{(x-a)^2+(y-b)^2}}$$

provided that the limit exists. Note that $ds = \sqrt{dx^2+dy^2}$ is the length of the vector $< dx, dy >$. We denote the line derivative by $\left.\frac{df(x,y)}{ds}\right|_{(a,b)}$ as usually.

Assume that $f(x,y)$ is line differentiable at (a,b), the line derivative of $f(x,y)$ at (a,b), which is a scaler, can be found further as follows

$$\left.\frac{df(x,y)}{ds}\right|_{(a,b)} = \left.\frac{df(x,y)}{< dx, dy >}\right|_{(a,b)} \cdot \frac{< dx, dy >}{ds} = < f_x(a,b), f_y(a,b) > \cdot < dx/ds, dy/ds >$$

It shows that the line derivative for a two-variable function $f(x,y)$ at (a,b), if it's line differentiable at (a,b), is equal to the line derivector of $f(x,y)$ at (a,b) inner product with the unit tangent vector $< dx/ds, dy/ds >$. that is, the projection of line derivector $< f_x(a,b), f_y(a,b) >$ to the unit vector $< dx/ds, dy/ds >$.

From the above fact, we are not hard,by the property of inner product, to know that the magnitude of the line derivector will be the largest slope of the function $f(x,y)$ at (a,b) and its direction will be the direction of the function that it has largest slope.

Remark 2.6. The line derivative is indeed what we called the Directional Derivative in traditional Calculus textbook.

Corollary 2.5. *The line derivative for a 3-variable function $f(x,y,z)$ at (a,b,c), if it's differentiable at (a,b,c), is equal to the line derivector of $f(x,y,z)$ at (a,b,c) inner product with the unit tangent vector $< dx/ds, dy/ds, dz/ds >$. That is, the projection of line derivector $< f_x(a,b,c), f_y(a,b,c), f(a,b,c) >$ to the unit vector $< dx/ds, dy/ds, dz/ds >$.*

Geometric interpretation of line derivative

The Line Derivative of $z = f(x,y)$ at (a,b) is a real number indicating the "monotonousness" of of the surface $z = f(x,y)$ at (a,b) in any given direction $< dx/ds, dy/ds >$. We use Line Derivative to define line (curve) integrals (For details see next section).

Theorem 2.6. *The function $f(x,y)$ is line differentiable at (a,b), if and only if*

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)-\nabla f(a,b)\cdot < x-a, y-b >}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

Proof. Since $f(x,y)$ is line differentiable at (a,b), from Theorem 2.1 we know that

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)}{< x-a, y-b >} =< f_x(a,b), f_y(a,b) > .$$

Thus

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,b)-<f_x(a,b),f_y(a,b)>\cdot<x-a,y-b>}{\sqrt{(x-a)^2+(y-b)^2}}=0$$

$$\Leftrightarrow \quad \lim_{<x,y>\to<a,b>}\left(\frac{f(x,y)-f(a,b)}{<x-a,y-b>}-<f_x(a,b),f_y(a,b)>\right)$$

$$\cdot \lim_{<x,y>\to<a,b>} \frac{<x-a,y-b>}{\sqrt{(x-a)^2+(y-b)^2}}=0$$

$$\Leftrightarrow \quad \left|\left(\lim_{<x,y>\to<a,b>}\left[\frac{f(x,y)-f(a,b)}{<x-a,y-b>}\right]-<f_x(a,b),f_y(a,b)>\right)\right.$$

$$\left.\cdot \lim_{<x,y>\to<a,b>} \frac{<x-a,y-b>}{\sqrt{(x-a)^2+(y-b)^2}}\right|=0$$

Since $\lim_{<x,y>\to<a,b>} \frac{<x-a,y-b>}{\sqrt{(x-a)^2+(y-b)^2}}$ is a unit vector, it implies

$$\lim_{<x,y>\to<a,b>}\left(\left[\frac{f(x,y)-f(a,b)}{<x-a,y-b>}\right]-<f_x(a,b),f_y(a,b)>\right)=<0,0>$$

$$\Leftrightarrow \quad \lim_{<x,y>\to<a,b>}\left[\frac{f(x,y)-f(a,b)}{<x-a,y-b>}\right]=<f_x(a,b),f_y(a,b)>$$

Thus we completed the proof of Theorem 2.6. □

Remark 2.7. Theorem 2.6 shows that "line differentiability" is equivalent to "differentiability" which is defined in traditional Advanced Calculus textbook.

Corollary 2.7. *If the function $f(x,y,z)$ is line differentiable at (a,b,c), then*

$$\lim_{<x,y,z>\to<a,b,c>} \frac{f(x,y,z)-f(a,b,c)-\nabla f(a,b,c)\cdot<x-a,y-b,z-c>}{\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}}=0$$

Theorem 2.8. *If both partial derivatives of $f(x,y)$ are continuous at (a,b), then $f(x,y)$ is line differentiable at (a,b), but not conversely.*

Proof. We need prove that the double limit $\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,y)}{x-a}$ exists. By the mean value theorem for 1-variable function, we have $\frac{f(x,y)-f(a,y)}{x-a}=f_x(\bar{x},y)$, where $\bar{x}$ is between a and x. Thus

$$\lim_{<x,y>\to<a,b>} \frac{f(x,y)-f(a,y)}{x-a}=\lim_{<x,y>\to<a,b>} f_x(\bar{x},y)=f_x(a,b)$$

since $f_x(x,y)$ and $f_y(x,y)$ are continuous at (a,b). Complete the proof. □

For "not conversely", see Example 2.1 in the following.

Remark 2.8. From Theorem 2.3, evidently that $f(x,y)$ is line differentiable at (a,b), it's only need that both its partial derivatives are partially continuous at (a,b),

Corollary 2.9. *If three partial derivatives of $f(x,y,z)$ are all continuous at (a,b,c), then $f(x,y,z)$ is line differentiable at (a,b,c), but not conversely.*

Theorem 2.10. *If $f(x,y)$ is line differentiable at (a,b), then $f(x,y)$ is continuous at (a,b), but not conversely.*

Proof. Consider the double limit, since $f(x,y)$ is line differentiable at (a,b), we have

$$\begin{aligned}
&\lim_{<x,y>\to<a,b>}(f(x,y)-f(a,b)) \\
= &\lim_{<x,y>\to<a,b>} < \frac{f(x,y)-f(a,y)}{x-a}, \frac{f(a,y)-f(a,b)}{y-b} > \cdot < x-a, y-b > \\
= &< f_x(a,b), f_y(a,b) > \cdot \lim_{<x,y>\to<a,b>} < x-a, y-b > \\
= &< f_x(a,b), f_y(a,b) > \cdot < 0,0 >= 0
\end{aligned}$$

Thus we complete the proof of Theorem 2.10. □

For "not conversely" part, see the following Example 2.2.

Remark 2.9. Notice that if $f(x,y)$ is continuous at (a,b), then $f(x,y)$ is partially continuous at (a,b), but not conversely.

Corollary 2.11. *If $f(x,y,z)$ is line differentiable at (a,b,c), then $f(x,y,z)$ is continuous at (a,b,c), but not conversely.*

Example 2.1. Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

then $f(x,y)$ is line differentiable at $(0,0)$, though both its partial derivatives are discontinuous at $(0,0)$, They are partially continuous at $(0,0)$.

Example 2.2. Let

$$f(x,y) = \begin{cases} (xy)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

then $f(x,y)$ is not line differentiable at $(0,0)$ and thus it is not differentiable at $(0,0)$ either.

Example 2.3. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

then

i) $\lim_{<x,y>\to<0,0>} \frac{f(x,y)-f(0,y)}{x-0}$ doesn't exist, but its partial derivatives at $(0,0)$; that is, $f_x(0,0)$, $f_y(0,0)$ exist.

ii) $f(x,y)$ is continuous at $(0,0)$, but it does not line differentiable at $(0,0)$.

Example 2.4. Let

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

then $f(x,y)$ is not line differentiable at $(0,0)$, (since both$f_x(0,y)$ and $f_y(x,0)$ are discontinuous at $(0,0)$).

iii) The plane derivative for a 2-variable function $f(x, y)$ at (a, b)

Let function $f(x, y)$ be defined on a plane convex region D containing a rectangular region with vertices $(a, b), (x, y), (a, y), (b, x)$. Define the plane derivative of $f(x, y)$ at (a, b) by the double limit

$$\lim_{<x,y>\to<a,b>} \frac{f(x, y) - f(x, b) - f(a, y) + f(a, b)}{(x - a)(y - b)}$$

provided that the double limit exists. We denote the plane derivative by $\left.\frac{d\breve{f}}{dA}\right|_{(a,b)} = \left.\frac{d\breve{f}(x,y)}{dxdy}\right|_{(a,b)}$, where $dA = dxdy$ and $\breve{f}(x, y) = f(x, y) - f(x, b) - f(a, y) + f(a, b)$.

Remark 2.10. Note that $\breve{f}(x, y)$ denotes the increment of $f(x, y)$ at (a, b) referring to the change of both variables x, y simultaneously. Since

$$f(x, y) - f(x, b) - f(a, y) + f(a, b) = [f(x, y) - f(a, b)] - [f(a, y) - f(a, b)] - [f(x, b) - f(a, b)]$$

Theorem 2.12. *Suppose that the function $\frac{f(x,y)-f(a,y)}{x-a}$ has limit $f_x(a, y)$ as $x \to a$ and $f_x(a, y)$ has partial derivative with respect to y at (a, b). If $f(x, y)$ has the plane derivative at (a, b), then*

$$\left.\frac{d\breve{f}(x, y)}{dA}\right|_{(a,b)} = \left.\frac{d\breve{f}(x, y)}{dxdy}\right|_{(a,b)} = f_{xy}(a, b).$$

Proof.

$$\begin{aligned}
\left.\frac{d\breve{f}(x, y)}{dxdy}\right|_{(a,b)} &\triangleq \lim_{<x,y>\to<a,b>} \frac{f(x, y) - f(a, y) - f(x, b) + f(a, b)}{(x - a)(y - b)} \\
&= \lim_{<x,y>\to<a,b>} \frac{\frac{f(x,y)-f(a,y)}{x-a} - \frac{f(x,b)-f(a,b)}{x-a}}{y - b} \\
&= \lim_{y\to b}[\lim_{x\to a} \frac{\frac{f(x,y)-f(a,y)}{x-a} - \frac{f(x,b)-f(a,b)}{x-a}}{y - b}] \\
&= \lim_{y\to b} \frac{f_x(a, y) - f_x(a, b)}{y - b} \\
&= f_{xy}(a, b)
\end{aligned}$$

□

From the Theorem 2.12, we know that the plane derivative of $f(x, y)$ at (a, b) is equal to taking the partial derivative with respect to $x(\text{or}y)$ first, then taking the partial derivative of the result with respect to $y(\text{or}x)$ next.

We say that $f(x, y)$ is plane differentiable at (a, b) if it has plane derivative $f_{xy}(a, b)$ at (a, b). Here, we need assume that $f(x, y) \in C^2$ and obviously we have $f_{yx}(a, b) = f_{xy}(a, b)$.

We use Plane Derivative to define plane integrals. (For details see next section)

Geometric interpretation of plane derivative

The Plane Derivative $f_{xy}(a, b)$ of surface $z = f(x, y)$ at (a, b) is a real number describing the instantaneous rate of the change of $f(x, y)$ at (a, b) to the area $dxdy$. It also indicating "non-flatness" of the surface $z = f(x, y)$ at (a, b) determined by four points $(a, b, f(a, b))$, $(a, y, f(a, y))$, $(x, b, f(x, b))$, $(x, y, f(x, y))$ with respect to the xy-coordinate plane.

iv) Surface derivectors for 3-variable functions

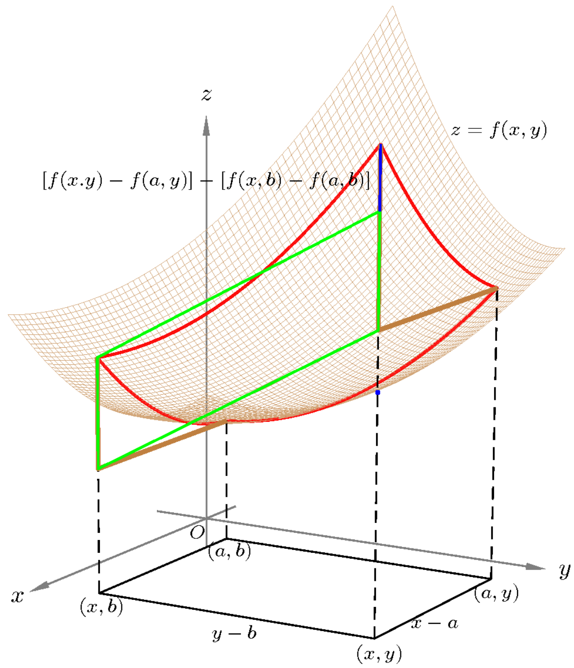

Figure 2.2: Geometric Interpretation of plane Derivative

Definition 2.3. (*Surface Derivector of the $f(x,y,z)$ at (a,b,c)*) *Surface derivector of the $f(x,y,z)$ at (a,b,c) is defined by triple limit*

$$\lim_{<x,y,z>\to<a,b,c>} \frac{f(x,y,z)-f(a,b,z)-f(x,b,c)-f(a,y,c)+2f(a,b,c)}{<(y-b)(z-c),(z-c)(x-a),(x-a)(y-b)>},$$

provided that the triple limit exists.

We denote the triple limit by $\frac{d\breve{\mathbf{f}}(\mathbf{x},\mathbf{y},\mathbf{z})}{<dydz,dzdx,dxdy>}\Big|_{(a,b,c)}$, if the triple limit exists. Notice that

$$\begin{aligned}\breve{f}(x,y,z) &\triangleq f(x,y,z)-f(a,b,z)-f(x,b,c)-f(a,y,c)+2f(a,b,c)\\ &= [f(x,y,z)-f(a,b,c)]-[f(x,b,c)-f(a,b,c)]-[f(a,y,c)-f(a,b,c)]\\ &\quad -[f(a,b,z)-f(a,b,c)]\end{aligned}$$

Remark 2.11. Analogously to line derivector of two-variable functions, we define from the equality

$$\begin{aligned}&\frac{f(x,y,z)-f(a,b,z)-f(x,b,c)-f(a,y,c)+2f(a,b,c)}{<(y-b)(z-c),(z-c)(x-a),(x-a)(y-b)>}\\ \triangleq\ &< \frac{f(x,y,z)-f(x,b,z)-f(x,y,c)+f(x,b,c)}{(y-b)(z-c)}, \frac{f(x,b,z)-f(a,b,z)-f(x,b,c)+f(a,b,c)}{(z-c)(x-a)},\\ &\frac{f(x,y,c)-f(a,b,z)-f(x,b,z)+f(a,b,c)}{((x-a)(y-b)} > \cdot <(y-b)(z-c),(z-c)(x-a),(x-a)(y-b)>\end{aligned}$$

as

$$\begin{aligned}
&\frac{f(x,y,z)-f(a,b,z)-f(x,b,c)-f(a,y,c)+2f(a,b,c)}{<(y-b)(z-c),(z-c)(x-a),(x-a)(y-b)>}\\
= & <\frac{f(x,y,z)-f(x,b,z)-f(x,y,c)+f(x,b,c)}{(y-b)(z-c)},\frac{f(x,b,z)-f(a,b,z)-f(x,b,c)+f(a,b,c)}{(z-c)(x-a)},\\
&\frac{f(x,y,c)-f(a,b,z)-f(x,b,z)+f(a,b,c)}{((x-a)(y-b)}>
\end{aligned}$$

We say that the function $f(x,y,z)$ is surface differentiable at (a,b,c) if it has surface derivector at (a,b,c). Here, we need assume that $f(x,y,z)\in C^2$. Then obviously we have

$$f_{xy}(a,b,c)=f_{yx}(a,b,c), f_{xy}(a,b,c), f_{zx}(a,b,c)=f_{xz}(a,b,c),\quad \text{etc.}$$

Theorem 2.13. *Assume that $f_{yz}(x,b,c)$, $f_{xz}(a,y,c)$ and $f_{xy}(a,b,z)$ are continuous at(a,b,c). If $f(x,y,z)$ is surface differentiable at (a,b,c), then*

$$\left.\frac{d\breve{f}(x,y,z)}{<dydz,dzdx,dxdy>}\right|_{(a,b,c)}=<f_{yz}(a,b,c),f_{zx}(a,b,c),f_{xy}(a,b,c)>$$

Proof. The triple limit

$$\begin{aligned}
&\lim_{<x,y,z>\to<a,b,c>}\frac{f(x,y,z)-f(x,b,c)-f(a,y,c)-f(a,b,z)+2f(a,b,c)}{<(y-b)(z-c),(z-c)(x-a),(x-a)(y-b)>}\\
= & \lim_{<x,y,z>\to<a,b,c>}<\frac{f(x,y,z)-f(x,b,z)-f(x,y,c)+f(x,b,c)}{(y-b)(z-c)},\\
&\frac{f(x,b,z)-f(a,b,z)-f(x,b,c)+f(a,b,c)}{(z-c)(x-a)},\frac{f(x,y,c)-f(x,b,c)-f(a,y,c)+f(a,b,c)}{(x-a)(y-b)}>\\
= & <f_{yz}(a,b,c),f_{zx}(a,b,c),f_{xy}(a,b,c)>
\end{aligned}$$

by the definition of the plane derivatives. □

Geometric interpretation of surface derivector

The Surface Derivector of space surface $u=f(x,y,z)$ at (a,b,c) is a three dimensional vector $<f_{yz}(a,b,c),fzx(a,b,c),f_{xy}(a,b,c)>$ describing the "non-flatness" of the space surface $u=f(x,y,z)$ at (a,b,c) to the three yz-, zx-, xy-coordinate planes.

v) Surface derivatives

If $f(x,y,z)$ is surface differentiable at (a,b,c), then the surface derivative is defined by the triple limit

$$\lim_{<x,y,z>\to<a,b,c>}\frac{f(x,y,z)-f(a,b,z)-f(a,y,c)-f(x,b,c)+2f(a,b,c)}{\sqrt{((y-b)(z-c))^2+((z-c)(x-a))^2+((x-a)(y-b))^2}}$$

provided that the limit exists.

The surface derivative at (a,b,c) is a real number and will be denoted by $\left.\frac{d\breve{f}(x,y,z)}{d|\mathbf{S}|}\right|_{(a,b,c)}=\left.\frac{d\breve{f}(x,y,z)}{\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}}\right|_{(a,b,c)}$, where $d|\mathbf{S}|=\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}$ is the length of the vector $<dydz,dzdx,dxdy>$ and $\mathbf{S}$ denotes the oriented space surface of domain S.

Theorem 2.14. *If $f(x,y,z)$ is surface differentiable at (a,b,c), then*

$$\frac{d\breve{f}(x,y,z)}{\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}}\bigg|_{(a,b,c)}$$
$$= \; < f_{yz}(a,b,c), f_{zx}(a,b,c), f_{xy}(a,b,c) > \cdot \frac{< dydz, dzdx, dxdy >}{\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}}$$

Proof.

$$\begin{aligned}
\frac{d\breve{f}(x,y,z)}{d|\mathbf{S}|}\bigg|_{(a,b,c)} &= \frac{d\breve{f}(x,y,z)}{\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}}\bigg|_{(a,b,c)} \\
&= \frac{d\breve{f}(x,y,z)}{< dydz, dzdx, dxdy >}\bigg|_{(a,b,c)} \cdot \frac{< dydz, dzdx, dxdy >}{\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}} \\
&= \; < f_{yz}(a,b,c), f_{zx}(a,b,c), f_{xy}(a,b,c) > \cdot \frac{< dydz, dzdx, dxdy >}{\sqrt{(dydz)^2+(dzdx)^2+(dxdy)^2}}
\end{aligned}$$

□

Thus we know that the surface derivative equals to the surface derivector inner product with a unit vector in the space surface $u = f(x,y,z)$ at (a,b,c); that is, the projection of surface derivector $< f_{yz}(a,b,c), f_{zx}(a,b,c), f_{xy}(a,b,c) >$ to the unit vector $< dydz/d|\mathbf{S}|, dzdx/d|\mathbf{S}|, \mathbf{dxdy/d|S|} >$.

Geometric interpretation of surface derivative

The Surface Derivative of space surface $u = f(x,y,z)$ at (a,b,c) is a real number

$$< f_{yz}(a,b,c), f_{zx}(a,b,c), f_{xy}(a,b,c) > \cdot < dydz/d|\mathbf{S}|, dzdx/d|\mathbf{S}|, dxdy/d|\mathbf{S}| >$$

describing the "non-planeness" of the space surface $u = f(x,y,z)$ at (a,b,c) to the three yz-, zx-, xy-coordinate planes in any given direction $< dydz/d|\mathbf{S}|, dzdx/d|\mathbf{S}|, dxdy/d|\mathbf{S}| >$.

We use Surface Derivative to define surface integrals. (For details see next section)

vi) Solid derivatives

The solid derivative at (a,b,c) is denoted by $\frac{d\breve{\breve{f}}(x,y,z)}{dV}\Big|_{(a,b,c)} = \frac{d\breve{\breve{f}}(x,y,z)}{dxdydz}\Big|_{(a,b,c)}$ and is defined by the triple limit $\lim_{<x,y,z>\to<a,b,c>} \frac{\breve{\breve{f}}(x,y,z)}{(x-a)(y-b)(z-c)}\Big|_{(a,b,c)}$ provided that the limit exists. Notice that

$$\begin{aligned}
\breve{\breve{f}}(x,y,z) &= f(x,y,z) - f(a,y,z) - f(x,b,z) - f(x,y,c) \\
&\quad + f(x,b,c) + f(a,y,c) + f(a,b,z) - f(a,b,c)
\end{aligned}$$

and since

$$\begin{aligned}
&f(x,y,z) - f(a,y,z) - f(x,b,z) - f(x,y,c) + f(x,b,c) + f(a,y,c) + f(a,b,z) - f(a,b,c) \\
= \; &[f(x,y,z) - f(a,b,c)] - [f(a,y,z) - f(a,b,c)] - [f(x,b,z) - f(a,b,c)] - [f(x,y,c) - f(a,b,c)] \\
&+ [f(x,b,c) - f(a,b,c)] + [f(a,y,c) - f(a,b,c)] + [f(a,b,z) - f(a,b,c)]
\end{aligned}$$

it denotes the increment of the function $f(x,y,z)$ as the three variables x,y,z change simultaneously. The solid derivative is obviously a real number, and is denoted by $\frac{d\breve{\breve{f}}(x,y,z)}{dV}\Big|_{(a,b,c)} = \frac{d\breve{\breve{f}}(x,y,z)}{dxdydz}\Big|_{(a,b,c)}$.

We say that $f(x,y,z)$ is solid differentiable at (a,b,c) if it has solid derivative at (a,b,c).

Theorem 2.15. *If $f(x,y,z)$ is solid differentiable at (a,b,c), then*

$$\left.\frac{d\breve{\breve{f}}(x,y,z)}{dxdydz}\right|_{(a,b,c)} = f_{xyz}(a,b,c)$$

Proof. Let $F(x,y,z) = f(x,y,z) - f(x,y,c) - f(x,b,z) + f(x,b,c)$. Using MVT of 1-variable function, we have

$$\begin{aligned} F(x,y,z) - F(a,y,z) &= [f(x,y,z) - f(x,y,c) - f(x,b,z) + f(x,b,c)] \\ &\quad - [f(a,y,z) - f(a,y,c) - f(a,b,z) + f(a,b,c)] \\ &= F_x(\bar{x},y,z)(x-a) \\ &= [f_x(\bar{x},y,z) - f_x(\bar{x},y,c) - f_x(\bar{x},b,z) + f_x(\bar{x},b,c)](x-a) \end{aligned}$$

Next let $\Psi(\bar{x},y,z) = f_x(\bar{x},y,z) - f_x(\bar{x},y,c) - f_x(\bar{x},b,z) + f_x(\bar{x},b,c)$, then

$$[\Psi_x(\bar{x},y,z) - \Psi_y(\bar{x},b,z)](x-a) = \Psi_{xy}(\bar{x},\bar{y},z)(y-b)(x-a),$$

i.e.

$$\begin{aligned} & [f(x,y,z) - f(x,y,c) - f(x,b,z) + f(x,b,c)] - [f(a,y,z) - f(a,y,c) - f(a,b,z) + f(a,b,c)] \\ = & [f_{xy}(\bar{x},\bar{y},z) - f_{xy}(\bar{x},\bar{y},c)](y-b)(x-a) \end{aligned}$$

thus

$$\begin{aligned} & [f(x,y,z) - f(x,y,c) - f(x,b,z) + f(x,b,c)] - [f(a,y,z) - f(a,y,c) - f(a,b,z) + f(a,b,c)] \\ = & f_{xyz}(\bar{x},\bar{y},\bar{z})(x-a)(y-b)(z-c) \end{aligned}$$

where $\bar{x}$ is between a,x;$\bar{y}$ is between b,y;$\bar{z}$ is between c,z. Taking limits as $(x,y,z) \to (a,b,c)$, we complete the proof. Here we need to assume that $f \in \mathbf{C^3}$, and thus $f_{yzx}(x,y,z) = f_{xyz}(x,y,z)$, etc.. □

Geometric interpretation of solid derivative

The Solid Derivative $f_{xyz}(a,b,c)$ of $u = f(x,y,z)$ at (a,b,c) is a real number describing the "non-rectangularness" of the space surface $u = f(x,y,z)$ at (a,b,c) determined by eight points $(a,b,c,f(a,b,c))$, $(a,y,z,f(a,y,z))$, $(x,b,z,f(x,b,z))$, $(x,y,c,f(x,y,c))$, $\cdots$, $(x,y,z,f(x,y,z))$ to the xyz-coordinate space.

2.2 The chain rules for differentiation of multi-variables functions

There are two types of the chain rules of line differentiations for multiple-variable functions:
Type I. $f(x,y), x = g(t),\ y = h(t)$: suppose that the function f is differentiable, then

$$\frac{df(x,y)}{dt} = f_x(x,y)g'(t) + f_y(x,y)h'(t) = < f_x(x,y), f_y(x,y) > \cdot < g'(t), h'(t) > .$$

Proof. Since f is differentiable

$$\begin{aligned} \frac{df(x,y)}{dt} &= \frac{df(x,y)}{< dx, dy >} \cdot \frac{< dx, dy >}{dt} = < f_x, f_y > \cdot < \frac{dx}{dt}, \frac{dy}{dt} > \\ &= < f_x, f_y > \cdot < g'(t), h'(t) > = f_x g'(t) + f_y h'(t) \end{aligned}$$

□

Type II. $f(x, y), x = g(u, v), y = h(u, v)$: suppose that the functions f, g, h are differentiable, then

$$\frac{df(x,y)}{<du,dv>} = \begin{bmatrix} f_x g_u + f_y h_u \\ f_x g_v + f_y h_v \end{bmatrix}.$$

Proof.

$$\begin{aligned}
\frac{df(g(u,v),h(u,v))}{<du,dv>} &= \frac{df(x,y)}{<dx,dy>} \cdot \frac{<dx,dy>}{<du,dv>} \\
&= <f_x(x,y), f_y(x,y)> \begin{bmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{bmatrix} \\
&= \begin{bmatrix} f_x g_u + f_y h_u \\ f_x g_v + f_y h_v \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\frac{<dx,dy>}{<du,dv>} &= \frac{<dg(u,v),dh(u,v)>}{<du,dv>} = <\frac{dg(u,v)}{<du,dv>}, \frac{dh(u,v)}{<du,dv>}> \\
&= \begin{bmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{bmatrix} = \begin{bmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{bmatrix}
\end{aligned}$$

□

Remark 2.12.

1. No chain rule for plane differentiation $\frac{d\breve{f}(x(u,v),y(u,v))}{dudv} \neq \frac{d\breve{f}(x,y)}{dxdy}\frac{dxdy}{dudv}$ while

$$\begin{aligned}
\breve{f}(x(u,v),y(u,v)) &= f(x(u,v)+\Delta u, y(u,v)+\Delta v) - f(x(u,v)+\Delta u, y(u,v)) \\
&\quad - f(x(u,v), y(u,v)+\Delta v) + f(x(u,v),y(u,v))
\end{aligned}$$

   and

$$\breve{f}(x,y) = f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) - f(x, y+\Delta y) + f(x,y)$$

   Then $\breve{f}(x(u,v),y(u,v)) \neq \breve{f}(x,y)$. Note also that $dx = (x_u du + x_v dv), dy = (y_u du + y_v dy)$, $\frac{dudu}{dudv} = \frac{du}{dv} = 0$, and $dvdu = dudv$. (due to u, v are free variables)

2. No chain rule for surface differentiation

$$\frac{d\breve{f}(x(u,v,w),y(u,v,w),z(u,v,w))}{<dvdw,dwdu,dudv>} \neq \frac{d\breve{f}(x,y,z)}{<dydz,dzdx,dxdy>} \cdot \frac{<dydz,dzdx,dxdy>}{<dvdw,dwdu,dudv>}$$

   Similar reason as the case of plane differentiation.

3. No chain rule for solid differentiation

$$\frac{d\breve{\breve{f}}(x(u,v,w),y(u,v,w),z(u,v,w))}{dudvdw} \neq \frac{d\breve{\breve{f}}(x,y,z)}{dxdydz}\frac{dxdydz}{dudvdw}$$

   Similar reason as the case of plane differentiation.

2.3 Differentials

There are two kinds of differentials for 2-variable functions: linear differentials and planar differentials, and three kinds of differentials for 3-variable functions: linear differentials, surface differentials and solid differentials.

i) Line differentials

Definition 2.4. *The line differential of a 2-variable function* $f(x,y)$ *at* (a,b) *is defined as*

$$df((a,b);ds) \triangleq < f_x(a,b), f_y(a,b) > \cdot < dx/ds, dy/ds > ds = f_x(a,b)dx + f_y(a,b)dy$$

Remark 2.13. The line differential of a 2-variable function $f(x,y)$ at (a,b) is also defined as $df((a,b);dx,dy) =< f_x(a,b), f_y(a,b) > \cdot < dx, dy >$.

Definition 2.5. *The line differential of a 3-variable function* $f(x,y,z)$ *at* (a,b,c) *is defined as*

$$\begin{aligned} df((a,b,c);ds) &\triangleq < f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) > \cdot < dx/ds, /dy/ds, dz/ds > ds \\ &= f_x(a,b,c)dx + f_y(a,b,c)dy + f_z(a,b,c)dz \end{aligned}$$

ii) Plane differentials

Definition 2.6. *The planar differential of a 2-variable function* $f(x,y)$ *at* (a,b) *is defined as*

$$d\breve{f}((a,b);dxdy) \triangleq f_{xy}(a,b)dxdy$$

iii) Surface differentials

Definition 2.7. *The surface differential of a 3-variable function* $f(x,y,z)$ *at* (a,b,c) *is defined as*

$$\begin{aligned} & d\breve{f}((a,b,c);dydz,dzdx,dxdy) \\ \triangleq\ & < f_{yz}(a,b,c), f_{zx}(a,b,c), f_{xy}(a,b,c) > \cdot < dydz, dzdx, dxdy > \\ =\ & f_{yz}(a,b,c)dydz + f_{zx}(a,b,c)dzdx + f_{xy}(a,b,c)dxdy \end{aligned}$$

iv) Solid differentials

Definition 2.8. *The solid differential of a 3-variable function* $f(x,y,z)$ *at* (a,b,c) *is defined as*

$$d\breve{\breve{f}}((a,b,c);dxdydz) \triangleq f_{xyz}(a,b,c)dxdydz$$

and the total differential of a 3-variable function $f(x,y,z)$ *at* (a,b,c) *is defined as*

$$\begin{aligned} & df((a,b,c);dx,dy,dz,dydz,dzdx,dxdy,dxdydz) \\ \triangleq\ & f_x(a,b,c)dx + f_y(a,b,c)dy + f_z(a,b,c)dz + f_{yz}(a,b,c)dydz \\ & + f_{zx}(a,b,c)dzdx + f_{xy}(a,b,c)dxdy + f_{xyz}(a,b,c)dxdydz \end{aligned}$$

Remark 2.14. We exploit those differentials to define multiple integrals (for details see next section).

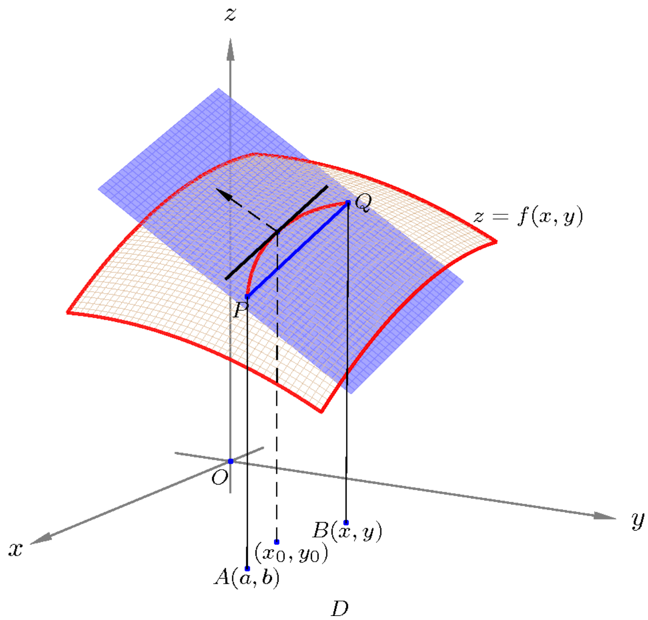

Figure 2.3: The Mean Vector Theorem

2.4 The mean vector theorems (MVTs)

i) MVTs: Under suitable conditions, we have several different types of MVTs.

Theorem 2.16 (The Line Mean Vector Theorem of 2-variable functions(LMVT))**.** *Let $f(x, y)$ be a line differentiable function on convex planar region D, and $(a, b) \in D$, then for all $(x, y) \in D$ there exists a point $(x_o, y_o) \in D$ such that*

$$f(x, y) - f(a, b) = < f_x(x_o, y_o), f_y(x_o, y_o) > \cdot < x - a, y - b >$$

Proof. We seek a point$(x_o, y_o) \in D$ such that its normal vector (to the surface $z = f(x, y)$) has the property

$$< f_x(x_o, y_o), f_y(x_o, y_o), -1 > \perp < x - a, y - b, f(x, y) - f(a, b) >$$

the normal vector of tangent plane of the surface $z = f(x, y)$ at $(x_o, y_o, f(x_o, y_o))$ is perpendicular to the line segment PQ in the plane Φ containing the line segment PQ and the arc of the surface $z = f(x, y)$ connecting PQ). Then one can always find a point on the arc, by drawing a tangent line paralleled to the segment PQ on the same plane Φ, until it touch the arc PQ. (See Figure 2.3.) Thus we have

$$< f_x(x_o, y_o), f_y(x_o, y_o), -1 > \perp < x - a, y - b, f(x, y) - f(a, b) >$$

That is,

$$f(x, y) - f(a, b) = < f_x(x_o, y_o), f_y(x_o, y_o) > \cdot < x - a, y - b >$$

then the point $(x_o, y_o) \in D$ will be the point we seek. We complete the proof of MVT. (The projection of the touch point $(x_o, y_o, f(x_o, y_o))$ (it always exists) to the xy-coordinate) □

Remark 2.15. Notice that the point (x_o, y_o) does not need on the line segment AB now but still $(x_0, y_0) \to (a, b)$ as $Q \to P$.

Theorem 2.17 (The Plane Mean Value Theorem for 2-variable functions)**.** *Let $f(x,y) \in C^2$ on convex region D, and $(a,b) \in D$, then $\forall (x,y) \in D$*

$$f(x,y) - f(a,y) - f(x,b) + f(a,b) = f_{xy}(\bar{x},\bar{y})(x-a)(y-b)$$

where $\bar{x}$ is between a, x and $\bar{y}$ is between b, y, respectively.

Proof. Let $F(t) = f(t,y) - f(t,b)$. Then using MVT for 1-variable function on $F(t)$, we get $[f_x(\bar{x},y) - f_x(\bar{x},b)](x-a)$. Then using MVT for 1-variable function again, we complete proof. □

Theorem 2.18 (The combined MVT for 2-variable function $f(x,y)$)**.** *If $f(x,y) \in C^2$, then*

$$f(x,y) - f(a,b) = f_x(\bar{x},b)(x-a) + f_y(a,\bar{y})(y-b) + f_{xy}(\bar{x},\bar{y})(x-a)(y-b)$$

where $\bar{x}$ are between a, x and $\bar{x}$ is between a, x and $\bar{y}$ is between b, y.

Proof. By use the MVT of 1-variable function with respect to x and y, we have

$$\begin{aligned} f(x,y) - f(a,b) &= f(x,y) - f(a,y) + f(a,y) - f(a,b) \\ &= f_x(\bar{x},y)(x-a) + f(a,y) - f((a,y) \\ &= [f_x(\bar{x},y) - f_x(\bar{x},b)](x-a) + f_x(\bar{x},b)(x-a) + f_y(a,\bar{y})(y-b) \\ &= f_{xy}(\bar{x},\bar{y})(y-b)(x-a) + f_x(\bar{x},b)(x-a) + f_y(a,\bar{y})(y-b) \end{aligned}$$

by letting $F(y) = [f_x(\bar{x},y) - f_x(\bar{x},b)](x-a) + f(a,y) - f(a,b)$, then using MVT to $F(y)$, we can get the same $\bar{y}$ in the expression where $\bar{x}$ is between a, x and $\bar{y}$ is between b, y. □

Theorem 2.19 (The Line Mean Vector Theorem for 3-variable function)**.** *Let $f(x,y,z)$ be function having continuous first partial derivatives on a convex region R and $(a,b,c) \in S$, then*

$$f(x,y,z) - f(a,b,z) =< f_x(\bar{x},\bar{y},\bar{z}), f_y(\bar{x},\bar{y},\bar{z}), f_z(\bar{x},\bar{y},\bar{z}) > \cdot < x-a, y-b, z-c >$$

where $(\bar{x},\bar{y},\bar{z})$ exist on the line segment determined by two points (a,b,c), (x,y,z).

Proof. Similar to the proof of Theorem 2.16. □

Theorem 2.20 (The Surface Mean Vector Theorem for 3-variable function(SMVT))**.** *Let $f(x,y,z)$ be function having continuous second partial derivatives on a convex region S and $(a,b,c) \in S$, then*

$$\begin{aligned} & f(x,y,z) - f(a,b,z) - f(x,b,c) - f(a,y,c) + 2f(a,b,c) \\ = \; & < f_{yz}(x,\bar{y},\bar{z}), f_{zx}(\bar{x},b,\bar{z}), f_{xy}(\bar{x},\bar{y},c) > \cdot < (y-b)(z-c), (z-c)(x-a), (x-a)(y-b) > \end{aligned}$$

where $(a,b,c) \in S, \bar{x}$ is between a, x; $\bar{y}$ is between b, y; $\bar{z}$ is between c, z and $\forall (x,y,z) \in S$.

Proof.

$$\begin{aligned} & f(x,y,z) - f(a,b,z) - f(x,b,c) - f(a,y,c) + 2f(a,b,c) \\ = \; & [f(x,y,z) - f(x,b,z) - f(x,y,c) + f(x,b,c)] + [f(x,b,z) - f(x,b,c) \\ & - f(a,b,z) + f(a,b,c)] + [f(x,y,c) - f(a,y,c) - f(x,b,c) + f(a,b,c)] \\ = \; & [f_y(x,\bar{y},z) - f_y(x,\bar{y},c)](y-b) + [f_z(x,b,\bar{z}) - f_z(a,b,\bar{z})](z-c) \\ & + [f_x(\bar{x},y,c) - f_x(\bar{x},b,c)](x-a) \\ = \; & f_{yz}(x,\bar{y},\bar{z})(z-c)(y-b) + f_{zx}(\bar{x},b,\bar{z})(x-a)(z-c) + f_{xy}(\bar{x},\bar{y},c)(y-b)(x-a) \\ = \; & < f_{yz}(x,\bar{y},\bar{z}), f_{zx}(\bar{x},b,\bar{z}), f_{xy}(\bar{x},\bar{y},c) > \cdot < (y-b)(z-c), (z-c)(x-a), (x-a)(y-b) > \end{aligned}$$

This completes the proof. □

Theorem 2.21 (The Mean Value Theorem of the Solid derivatives for 3-variable function)**.** *Let $f(x,y,z)$ be a function having continuous third partial derivatives on a convex region E and $(a,b,c) \in E$, then*

$$\begin{aligned} & f(x,y,z)-f(a,y,z)-f(x,b,z)-f(x,y,c)+f(x,b,c)+f(a,y,c)+f(a,b,z)-f(a,b,c) \\ = \ & f_{xyz}(\bar{x},\bar{y},\bar{z})(x-a)(y-b)(z-c) \end{aligned}$$

where $(a,b,c) \in E, \bar{x}$ is between a, x; $\bar{y}$ is between b, y; $\bar{z}$ is between c, z and $\forall (x,y,z) \in E$.

Proof. Using MVT for 1-variable function

$$\begin{aligned} & f(x,y,z)-f(a,y,z)-f(x,b,z)-f(x,y,c)+f(x,b,c)+f(a,y,c)+f(a,b,z)-f(a,b,c) \\ = \ & [f(x,y,z)-f(a,y,z)-f(x,b,z)+f(a,b,z)]-[f(x,y,c)-f(a,y,c)-f(x,b,c)+f(a,b,z)] \\ = \ & [f_x(\bar{x},y,z)-f_x(\bar{x},b,z)](x-a)-[f_x(\bar{x},y,c)-f_x(\bar{x},b,c)](x-a) \\ = \ & [f_{xy}(\bar{x},\bar{y},z)-f_{xy}(\bar{x},\bar{y},c)](y-b)(x-a) \\ = \ & f_{xyz}(\bar{x},\bar{y},\bar{z})(x-a)(y-b)(z-c) \end{aligned}$$

□

Theorem 2.22 (The combined MVT for 3-variable function $f(x,y,z)$)**.** *If $f(x,y,z) \in C^3$, then*

$$\begin{aligned} & f(x,y,z)-f(a,b,c) \\ = \ & f_x(\bar{x},b,c)(x-a)+f_y(a,\bar{y},c)(y-b)+f_z(a,b,\bar{z})(z-c)+f_{yz}(a,\bar{y},\bar{z})(y-b)(z-c) \\ & +f_{zx}(\bar{x},b,\bar{z})(z-c)(x-a)+f_{xy}(\bar{x},\bar{y},c)(x-a)(y-b) \\ & +f_{xyz}(\bar{x},\bar{y},\bar{z})(x-a)(y-b)(z-c) \end{aligned}$$

where $\bar{x}$ is between a, x; $\bar{y}$ is between b, y; $\bar{z}$ is between c, z.

Proof. By use the MVT of 1-variable function

$$\begin{aligned} & f(x,y,z)-f(a,b,c) \\ = \ & f(x,y,z)-f(a,y,z)+f(a,y,z)-f(a,b,z)+f(a,b,z)-f(a,b,c) \\ = \ & f_x(\bar{x},y,z)(x-a)+f_y(a,\bar{y},z)(y-b)+f_z(a,b,\bar{z})(z-c) \\ = \ & [f_{xy}(\bar{x},\bar{y},z)(y-b)](x-a)+f_x(\bar{x},b,z)(x-a)+f_y(a,\bar{y},z)(y-b)+f_z(a,b,\bar{z})(z-c) \\ = \ & [f_{xy}(\bar{x},\bar{y},z)-f_{xy}(\bar{x},\bar{y},c)](y-b)(x-a)+f_{xy}(\bar{x},\bar{y},c)(y-b)(x-a) \\ & +[f_x(\bar{x},b,z)(x-a)-f_x(\bar{x},b,c)(x-a)]+[f_y(a,\bar{y},z)(y-b)-f_y(a,\bar{y},c)(y-b)] \\ & +f_y(a,\bar{y},c)(y-b)+f_x(\bar{x},b,c)(x-a)+f_z(a,b,\bar{z})(z-c) \\ = \ & f_{xyz}(\bar{x},\bar{y},\bar{z})(z-c)(y-b)(x-a)+f_{xy}(\bar{x},\bar{y},c)(y-b)(x-a)+f_{xz}(\bar{x},b,\bar{z})(z-c)(x-a) \\ & +f_{yz}(a,\bar{y},\bar{z})(z-c)(y-b)+f_y(a,\bar{y},c)(y-b)+f_x(\bar{x},b,c)(x-a)+f_z(a,b,\bar{z})(z-c) \end{aligned}$$

where $\bar{x}$ is between a, x; $\bar{y}$ is between b, y; $\bar{z}$ is between c, z. □

3 Integrations

3.1 Definitions of various multiple integrals

There are also many types of integrals in multiple integral: Line integrals, Plane integrals, Surface integrals and Solid integrals, each constructed corresponding to their differentials.
i) Line integrals for $f(x,y)$

Definition 3.1. *If we take the integral of line differentials for $f(x, y)$ along the oriented plane curve* $\mathbf{C}$*.*

$$\int df((x, y); dx, dy) \triangleq \int_{\mathbf{C}} < f_x(x, y), f_y(x, y) > \cdot < dx, dy >$$

over the oriented plane curve $\mathbf{C}$*, then the integral is called the line integral of $f(x, y)$ over* $\mathbf{C}$*. We denote this line integral by $\int_{\mathbf{C}} f_x(x, y)dx + f_y(x, y)dy$, where $\mathbf{C}$ denotes an oriented plane curve connecting the variables x, y.*

Remark 3.1. Same as in the single integral of $f(x)$, we can treat line integral as anti-derivative of line derivative for function $f(x, y)$ over oriented plane curve $\mathbf{C}$.

Definition 3.2 (Line Integrals for $f(x, y, z)$)**.** *If we take the integral of line differentials for $f(x, y)$ along the oriented space curve* $\mathbf{C}$

$$\int df((x, y, z); dx, dy, dz) \triangleq \int_{\mathbf{C}} < f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) > \cdot < dx, dy, dz >$$

over the oriented space curve $\mathbf{C}$ (provided the integral exists), then the integral is called the line integral of $f(x, y, z)$ over $\mathbf{C}$*. We denote this line integral by $\int_{\mathbf{C}} f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$, where $\mathbf{C}$ denotes an oriented space curve connecting the variables x, y, z.*

ii) Plane integral for $f(x, y)$

Definition 3.3. *If we take the integral of plane differentials for $f(x, y)$ over oriented plane region* $\mathbf{D}$

$$\int_{\mathbf{D}} d(\breve{f}(x, y); dxdy) \triangleq \int_{\mathbf{D}} f_{xy}(x, y)dxdy$$

then the integral is called the plane integral of $f(x, y)$ over the oriented plane region $\mathbf{D}$. *We denote this plane integral by $\int_{\mathbf{D}} f_{xy}(x, y)dxdy$.*

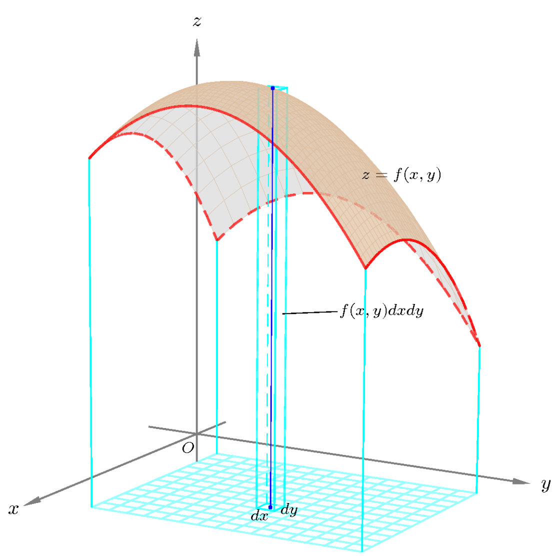

Figure 3.1: Plane Integral for $f(x, y)$

Remark 3.2.

1. Plane integrals are same as double integrals.
2. Same as in the single integral of $f(x)$, we can treat plane integral as anti-derivative of plane derivative for function $f(x, y)$ over oriented plane integral $\mathbf{D}$.

iii) Surface integral for $f(x, y, z)$ over oriented space surface region S

Definition 3.4. *If we take the integral of surface differentials for* $\mathbf{S}$

$$\begin{aligned} & \int_{\mathbf{S}} d(\breve{f}(x,y,z); dydz, dzdx, dxdy) \\ = & \int_{\mathbf{S}} [< f_{yz}(x,y,z), f_{zx}(x,y,z), f_{xy}(x,y,z) > \cdot < dydz/d\mathbf{S}, dzdx/d\mathbf{S}, dxdy/d\mathbf{S} >] d\mathbf{S} \\ = & \int_{\mathbf{S}} f_{yz}(a,b,c)dydz + f_{zx}(a,b,c)dzdx + f_{xy}(a,b,c)dxdy \end{aligned}$$

over the oriented space surface region $\mathbf{S}$*, then the integral is called the surface integral of* $f(x, y, z)$ *over* $\mathbf{S}$*. We denote this surface integral by*

$$\begin{aligned} \int_{\mathbf{S}} d(\breve{f}(x,y,z); dydz, dzdx, dxdy) &= \int_{\mathbf{S}} < f_{yz}(x,y,z), f_{zx}(x,y,z), f_{xy}(x,y,z) > d\mathbf{S} \\ &= \int_{\mathbf{S}} f_{yz}(x,y,z)dydz + f_{zx}x, y, zdzdx + f_{xy}(x,y,z)dxdy \end{aligned}$$

Note that $\mathbf{S}$ is an oriented smooth space surface in 3-dim space.

iv) Solid integral for $f(x, y, z)$

Definition 3.5. *If we take the integral of solid differentials for* $f(x, y, z)$

$$\breve{\breve{f}}((x,y,z); dxdydz) \triangleq f_{xyz}(a,b,c)dxdydz$$

over the oriented space box $\mathbf{B}$*, then the integral is called the solid integral of* $f(x, y, z)$ *over the oriented box* $\mathbf{B}$*. We denote this solid integral by* $\int_{\mathbf{B}} f_{xyz}(x, y, z)dxdydz$*.*

Remark 3.3. Same as in single integral, we can treat triple integral as anti-derivative of solid derivative for function $f(x, y, z)$. For the details see next section.

3.2 The fundamental properties of plane integrals and solid integrals

i) $f(x, y)$ is integrable function on $\mathbf{D}$

$$\int_{\mathbf{D}} f(x,y)dxdy = 0$$

where $\mathbf{D}$ is an oriented plane region.

Reasoning: since $dxdy$ does not represented an area element in taking plane integrals.

Similarly,

$$\int_{\mathbf{B}} f(x,y,z)dxdydz = 0$$

where $\mathbf{B}$ is an oriented space region.

ii) $f(x,y)$ is integrable function on $\mathbf{D}$

$$\int_{\mathbf{D}} f(x,y)dydx = -\int_{\mathbf{D}} f(x,y)dxdy$$

where $\mathbf{D}$ is an oriented plane region.

Reasoning: When $\mathbf{D}$ is oriented with counter-clock, $dydx$ will be oriented by the inverse direction to that of $dxdy$.

Similarly,

$$\begin{aligned}\int_{\mathbf{B}} f(x,y,z)dydxdz &= -\int_{\mathbf{B}} f(x,y,z)dxdydz, \cdots \\ \int_{\mathbf{B}} f(x,y,z)dzdxdy &= \int_{\mathbf{B}} f(x,y,z)dxdydz, \cdots\end{aligned}$$

for all integrable functions $f(x,y,z)$ on $\mathbf{B}$ and $\mathbf{B}$ is an oriented space region.

3.3 Various fundamental theorems of calculus for multiple integrals

Due to various integrals defined as previous section, we have corresponding Fundamental Theorems of Calculus for Multiple Integrals.

1. Fundamental Theorems of Calculus (FTC) for Line (Curve) Integrals

For 2-variable and 3-variable.

2. Fundamental Theorems of Calculus for Plane Integrals.
3. Fundamental Theorems of Calculus for Surface Integrals.
4. Fundamental Theorems of Calculus for Solid Integrals.

Theorem 3.1 (FTC for curve integral of $f(x,y)$ along the oriented curve $\mathbf{C}$)**.** *Consider the oriented curve* $\mathbf{C} : \boldsymbol{r}(t) =< x(t), y(t) >, t \in [\alpha, \beta]$*, the FTC for curve integral of* $f(x,y)$ *along the oriented curve* $\boldsymbol{C}$ *is*

$$\int_{\boldsymbol{C}} f_x(x,y)dx + f_y(x,y)dy = \int_{\alpha}^{\beta} df(\mathbf{r(t)}) = \mathbf{f(c,d)} - \mathbf{f(a,b)}$$

where $\mathbf{r}(\alpha) = (\mathbf{a}, \mathbf{b})$ *is the initial point of the oriented curve* $\boldsymbol{C}$*, and* $\mathbf{r}(\beta) = (\mathbf{c}, \mathbf{d})$ *is its terminal point.*

Remark 3.4. The line integral $\int_{\mathbf{C}} f_x(x,y)dx + f_y(x,y)dy$ can be also evaluate as

$$\int_{\mathbf{C}} f_x(x,b)dx + f_y(c,y)dy = [f(c,b) - f(a,b)] + [f(c,d) - f(c,b)]$$

(since the line integral is independent of path.)

Theorem 3.2 (FTC for curve integral of $f(x,y,z)$ along the oriented space curve $\mathbf{C}$)**.** *Consider the oriented space curve* $\mathbf{C} : \boldsymbol{r}(t) =< x(t), y(t), z(t) >, \alpha \leqq t \leqq \beta$*, the FTC for curve integral of* $f(x,y,z)$ *along the oriented space curve* $\mathbf{C}$ *is given by*

$$\begin{aligned}\int_{\mathbf{C}} f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz &= \int_{\mathbf{C}} df(x,y,z) = \int_{\mathbf{r}(\alpha)}^{\mathbf{r}(\beta)} df(x,y,z) \\ &= f(\mathbf{r}(\beta)) - \mathbf{f}(\mathbf{r}(\alpha)) = \mathbf{f}(\mathbf{d},\mathbf{e},\mathbf{f}) - \mathbf{f}(\mathbf{a},\mathbf{b},\mathbf{c})\end{aligned}$$

where $\mathbf{r}(\alpha) = (\mathbf{a}, \mathbf{b}, \mathbf{c})$ *is the initial point of* $\mathbf{C}$ *and* $\mathbf{r}(\beta) = (\mathbf{d}, \mathbf{e}, \mathbf{f})$ *is its terminal point.*

Illustration: By the anti-derivative of line derivector, we have

$$\int_{\mathbf{C}} f_x(x,y,z)dx + f_y(x,y,z)dy + f_z(x,y,z)dz \;=\; \int_{\mathbf{C}} df \;=\; f(x,y,z) - f(a,b,c)$$

Theorem 3.3 (FTC for plane integral of $f(x,y)$ over the plane rectangular region $\mathbf{D}$)**.** *Consider the plane rectangular region* $\mathbf{D} = [\mathbf{a},\mathbf{b}] \times [\mathbf{c},\mathbf{d}]$)*, the FTC for plane integral of* $f(x,y)$ *over the plane rectangular region* $\mathbf{D}$ *is*

$$\int_{\mathbf{D}} f_{xy}(x,y)dA \;=\; \iint_{\mathbf{D}} f_{xy}(x,y)dxdy \;=\; \int_{\mathbf{D}} d\breve{f} \;=\; f(b,d) - f(a,d) - f(b,c) + f(a,c)$$

by the anti-derivative of plane derivative.

Theorem 3.4 (FTC for surface integral)**.** *The FTC for surface integral of* $f_{yz}(x,y,z)dydz + f_{zx}(x,y,z)dzdx + f_{xy}(x,y,z)dxdy >$ *over the oriented space surface* $\boldsymbol{S}$ *is*

$$\begin{aligned}\int_{\boldsymbol{S}} \frac{d\breve{f}(x,y,z)}{d\boldsymbol{S}} \cdot d\boldsymbol{S} &= \int_{\boldsymbol{S}} d\breve{f}((x,y,z); dydz, dzdx, dxdy) \\ &= \int_{\mathbf{S}} f_{yz}(x,y,z)dydz + f_{zx}(x,y,z)dzdx + f_{xy}(x,y,z)dxdy \\ &= f(d,e,f) - f(a,e,c) - f(d,b,c) - f(a,b,f) + 2f(a,b,c)\end{aligned}$$

by the anti-derivative of surface derivative, where the surface $\boldsymbol{S}$ *is oriented by its outward normal vector, containing five points* (d,e,f), (a,e,c), (d,b,c), (a,b,f), (a,b,c) *and* $d\boldsymbol{S} =< dydz, dzdx, dxdy >$.

Remark 3.5. The surface integral $\int_{\mathbf{S}} f_{yz}(x,y,z)dydz + f_{zx}(x,y,z)dzdx + f_{xy}(x,y,z)dxdy$ can each be evaluated by plane integrals

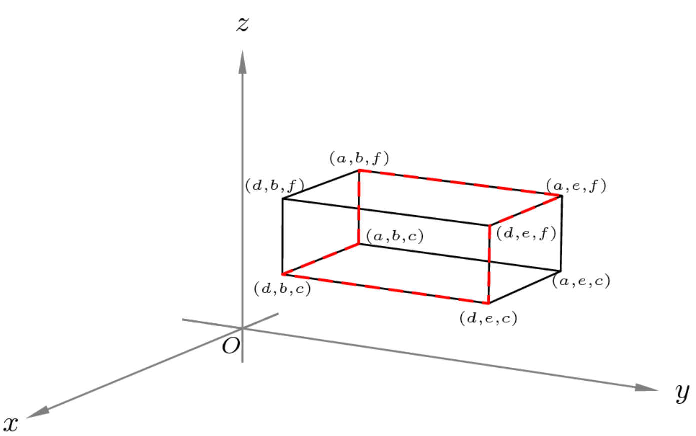

Figure 3.2: The surface integral

$$\begin{aligned} & \int_{\mathbf{S}} d(f(x,y,z); dydz, dzdx, dxdy) \\ = & \int_{\mathbf{S}} f_{yz}(d,y,z)dydz + f_{zx}(x,b,z)dzdx + f_{xy}f(x,y,c)dxdy \\ = & [f(d,e,f) - f(d,b,f) - f(d,e,c) + f(d,b,c)] + [f(d,b,f) - f(a,b,f) - f(d,b,c) + f(a,b,c)] \\ & +[f(d,e,c) - f(d,b,c) - f(a,e,c) + f(a,b,c)] \\ = & f(d,e,f) - f(a,e,c) - f(d,b,c) - f(a,b,f) + 2f(a,b,c)\end{aligned}$$

where **S** contains six points (d,e,f), (d,e,c), (d,b,c), (d,b,f), (a,b,f), (a,b,c). See the Figure 3.2.

Analogously to Theorem 3.2, we can obtain the above result.

Theorem 3.5 (FTC for solid integral of $f_{xyz}(x,y,z)$)**.** *The FTC for solid integral of $f_{xyz}(x,y,z)$ is given by*

$$\begin{aligned}
&\int_{\mathbf{E}}\left(\frac{d\breve{\breve{f}}(x,y,z)}{dV}\right)dV \\
=\ &\int_{\mathbf{E}} d\breve{\breve{f}}((x,y,z);dxdydz) \ = \ \iiint_{\mathbf{E}} f_{xyz}(x,y,z)dxdydz \\
=\ &f(x,y,z)-f(x,y,c)-f(x,b,z)-f(a,y,z)+f(a,b,z)+f(a,y,c)+f(x,b,c)-f(a,b,c) \\
=\ &[f(x,y,z)-f(a,b,c)]-[f(a,y,z)-f(a,b,c)]-[f(x,y,c)-f(a,b,c)]-[f(x,b,z)-f(a,b,c)] \\
&+[f(a,b,z)-f(a,b,c)]+[f(a,y,c)-f(a,b,c)]+[f(x,b,c)-f(a,b,c)]
\end{aligned}$$

where **E** *denotes the oriented rectangular box containing the following six points*

$$(x,y,z),(x,y,c),(x,b,z),(a,y,z),(a,b,z),(a,y,c),(x,b,c),(a,b,c)$$

as vertices, oriented with its normal vectors outwardly, and $dV = dxdydz$.

Analogously to Theorem 3.2, we are able to prove the above result.

Why we introduce those FTCs for multiple integrals? The main goal is to evaluate those multiple integrals listed in previous section.

From those FTCs,it's easy to obtain the following combined FTCs.

Combined FTCs for 2-variable function

Let $f(x,y)$ be a twice continuous differentiable function defined on a rectangular region D with $(a,b),(c,d)$ as diagonal vertices. Then

$$\int_a^b f_x(x,c)dx + \int_c^d f_y(a,y)dy + \int_a^b\int_c^d f_{xy}(x,y)dxdy = f(b,d)-f(a,c)$$

Combined FTC for 3-variable function

Let $f(x,y,z)$ $\mathbf{C^3}$ function defined on a rectangular solid region **E** with (a,b,c), (d,e,f) as diagonal vertices. Then

$$\begin{aligned}
&\int_a^b f_x(x,c,e)dx + \int_c^d f_y(a,y,e)dy + \int_e^f f_z(a,c,z)dz + \int_a^b\int_c^d f_{xy}(x,y,e)dxdy \\
&+\int_c^d\int_e^f f_{yz}(a,y,z)dydz + \int_a^b\int_e^f f_{zx}(x,c,z)dzdx + \int_a^b\int_c^d\int_e^f f_{xyz}(x,y,z)dxdydz \\
=\ &f(b,d,f)-f(a,c,e)
\end{aligned}$$

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古典多變函數微積分之新構思

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中文摘要

本論文是針對“多變數”(兩個以上)變數函數之導數而寫。長久以來多變數函數之導數均以偏導數代之，本論文則採取直接定義，這是以前不曾出現過的。因有了多變數函數之導數，其對應的多積分及各種有關多微積分的問題，也都可衍用單變數微積分的特性處理，使多微積分的問題變成簡單而易學。

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