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On the Convolution Property of the Poisson Kernels

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Abstract

Based on examples of the exact solutions of the two-dimensional Laplace's equation, I derive the convolution formulas among the Poisson kernels associated with given domains. These mathematical identities can be verified directly using the residue theorem in the complex analysis. The interconnections of convolution formulas for different domains are established via a conformal transformation.

Keywords : Laplace's equation, Poisson kernel, convolution formula, contour integral, residue theorem.

1 Introduction

It is well-known that the boundary-value problem of the Laplace's equation can be solved via a convolution formula of the Poisson kernel with a given boundary data,

$$\Phi(\vec{r}) = \int_{\partial\Omega} ds\ K_\Omega(\vec{r};\vec{r}'(s)) \cdot \Phi(\vec{r}'(s))|_{\vec{r}'(s)\in\partial\Omega}. \tag{1}$$

The explicit form of the Poisson kernel, $K_\Omega(\vec{r};\vec{r}')$, depends on the domain of the solution, Ω, and is independent of the boundary data. Consequently, we can view the convolution formula, Eq.(1), as a linear map from the space of admissible boundary datas to the harmonic function defined on Ω. In addition, there is a natural composition rule among these linear maps associated with the inclusions of domains ($\Omega_2 \subsetneq \Omega_1,\ \vec{r} \in int(\Omega_2),\ \vec{r}'' \in \partial\Omega_1$):

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$$K_{\Omega_1}(\vec{r};\vec{r}'') = \int_{\partial\Omega_2} ds\ K_{\Omega_2}(\vec{r};\vec{r}'(s)) \cdot K_{\Omega_1}(\vec{r}'(s);\vec{r}'')|_{\vec{r}'(s)\in\partial\Omega_2}. \tag{2}$$

In the following, Eq.(2) is referred to as a convolution formula of the Poisson kernels, which is the main subject of this paper.

To be specific, I shall consider the solutions of the two-dimensional Laplace's equation subject to two types of boundary conditions:

- Case I:
  In this case, we solve the Laplace's equation, $\Delta\Phi(x,y) = 0$, defined on the upper-half plane(UHP), H_2,

$$H_2 := \{(x,y) \mid x \in \mathbb{R},\ y \geq 0\}, \qquad \partial H_2 = \{(x,y) \mid x \in \mathbb{R},\ y = 0\}. \tag{3}$$

  The reduced Poisson kernel for H_2 is defined as

$$G_{H_2}(x,y) := \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right). \tag{4}$$

  The Poisson kernel for H_2 (referred to as the UHP Poisson kernel) is defined as

$$K_{H_2}(x,y;\xi,\eta) := G_{H_2}(x-\xi,y-\eta) = \frac{1}{\pi}\left[\frac{y-\eta}{(x-\xi)^2+(y-\eta)^2}\right]. \tag{5}$$

  The Laplace's equation on H_2 admits a unique solution, if we specify a boundary data, $f(\xi) := \Phi(\xi,0)$, at ∂H_2.

$$\Phi(x,y) = \int_{-\infty}^{\infty} d\xi\ G_{H_2}(x-\xi,y)\ f(\xi) = \int_{-\infty}^{\infty} d\xi\ K_{H_2}(x,y;\xi,0)\ f(\xi) \tag{6}$$

- Case II:
  In this case, we solve the Laplace's equation, $\Delta\Psi(\rho,\phi) = 0$, defined on a disk with radius R, D_2.

$$D_2 := \{(\rho,\phi) \mid 0 \leq \rho \leq R,\ -\pi \leq \phi \leq \pi\}, \tag{7}$$

  and the boundary of D_2 is a circle,

$$\partial D_2 = S^1 = \{(\rho,\phi) \mid \rho = R,\ -\pi \leq \phi \leq \pi\}. \tag{8}$$

  The Poisson kernel for D_2 (referred to as the disk Poisson kernel) is defined as

$$K_{D_2}(\rho,\phi;R,\chi) := \left[\frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\chi-\phi)}\right] \tag{9}$$

The Laplace's equation on D_2 admits a unique solution, if we specify a boundary value, $g(\chi) := \Psi(R,\chi)$ at ∂D_2.

$$\Psi(\rho,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi\; K_{D_2}(\rho,\phi;R,\chi)\; g(\chi). \tag{10}$$

The crucial importance of the Poisson kernels, Eqs.(5),(9), of the Laplace's equation lies in the following properties:

(1) The Poisson kernels are fundamental solutions of the Laplace's equation.

$$\text{Case I: } \Delta K_{H_2}(x,y;\xi,\eta) := \left(\partial_x^2+\partial_y^2\right) K_{H_2}(x,y;\xi,\eta) = 0, \tag{11}$$

$$\text{Case II: } \Delta K_{D_2}(\rho,\phi;R,\chi) := \left(\partial_\rho^2+\frac{1}{\rho}\partial_\rho+\frac{1}{\rho^2}\partial_\phi^2\right) K_{D_2}(\rho,\phi;R,\chi) = 0. \tag{12}$$

(2) The Poisson kernels approach to delta functions as the interior points approach to the boundaries of the domains.

$$\text{Case I}: \int_{-\infty}^{\infty} d\xi\; K_{H_2}(x,y;\xi,\eta) = 1, \qquad \lim_{y\to\eta} K_{H_2}(x,y;\xi,\eta) = \delta(x-\xi), \tag{13}$$

$$\text{Case II}: \int_{-\pi}^{\pi} d\chi\; K_{D_2}(\rho,\phi;R,\chi) = 2\pi, \qquad \lim_{\rho\to R} K_{D_2}(\rho,\phi;R,\chi) = 2\pi\delta(\phi-\chi). \tag{14}$$

The purpose of this paper is to discuss another property of the Poisson kernel, the convolution(evolution) property. This property is analogous to the evolution property of the heat kernel, K_T, of the thermal conduction equation for a temperature distribution $u(x,t)$,

$$\partial_t u = \alpha\Delta u \quad \Rightarrow \quad u(x,t) = \int_{-\infty}^{\infty} d\xi\; K_T(x,t;\xi,\tau)\; u(\xi,\tau), \tag{15}$$

$$K_T(x,t;\xi,\tau) := \sqrt{\frac{1}{4\pi\alpha(t-\tau)}}\exp\left[-\frac{(x-\xi)^2}{4\alpha(t-\tau)}\right], \tag{16}$$

$$K_T(x,t;z,t'') = \int_{-\infty}^{\infty} dy\; K_T(x,t;y,t')\cdot K_T(y,t';z,t''). \tag{17}$$

Similarly, the evolution operators, $U(t,t_0)$, of the Schrödinger equation for a quantum state $|\psi(t)\rangle$,

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle \quad \Rightarrow \quad i\hbar\frac{\partial}{\partial t}U(t,t_0) = H\cdot U(t,t_0), \tag{18}$$

$$U(x,t;y,t') := \langle x|\ \exp\left[-\frac{i}{\hbar}H(t-t')\right]\ |y\rangle, \tag{19}$$

obey the same composition rule,

$$U(x,t;z,t'') = \int_{-\infty}^{\infty} dy\ U(x,t;y,t')\cdot U(y,t';z,t''). \tag{20}$$

Thus, even though we usually consider the input variables of the solutions to the Laplace's equation as spatial coordinates, it is suggestive that one can identify one of the variables to be the "time" coordinate in the discussion of the convolution property.

This paper is organized as follows: I firstly identify a mathematical equation for the convolution formula of the Poisson kernel on H_2. Then I verify this identity based on direct computations in Sec.2. Similar discussions/computations are repeated in Sec.3 for the case of disk geometry, D_2. In Sec.4, I establish the connections between two geometries/equations based on a complex conformal transformation. This also leads to a more general type of convolution formula associated with the disk geometry, the details of which are given in App. A.

2 Poisson Kernel Associated with the Upper Half Plane (UHP)

2.1 Formulating the convolution formula on H_2

To facilitate the discussion, it is useful to express the convolution formula for the solutions of the Laplace's equation on H_2 in the following form,

$$\Phi(x_2,y_2) = \int_{-\infty}^{\infty} dx_1\ K_{H_2}(x_2,y_2;x_1,y_1)\ \Phi(x_1,y_1). \tag{21}$$

Similarly, by changing the coordinates, $y_1 \to y_2, y_2 \to y_3$, we obtain

$$\begin{aligned}\Phi(x_3,y_3) &= \int_{-\infty}^{\infty} dx_2\ K_{H_2}(x_3,y_3;x_2,y_2)\ \Phi(x_2,y_2) &(22)\\ &= \int_{-\infty}^{\infty} dx_1\ K_{H_2}(x_3,y_3;x_1,y_1)\ \Phi(x_1,y_1). &(23)\end{aligned}$$

Note that in Eqs.(21), (23), we have assumed that the results of both convolutions is independent of the choice of y_1, y_2. Now the convolution property of the Poisson kernel becomes apparent if we compose Eqs.(21),(22):

$$\begin{aligned}\Phi(x_3,y_3) &= \int_{-\infty}^{\infty} dx_2\ K_{H_2}(x_3,y_3;x_2,y_2)\left[\int_{-\infty}^{\infty} dx_1 K_{H_2}(x_2,y_2;x_1,y_1)\Phi(x_1,y_1)\right] \\ &= \int_{-\infty}^{\infty} dx_1\left[\int_{-\infty}^{\infty} dx_2 K_{H_2}(x_3,y_3;x_2,y_2)K_{H_2}(x_2,y_2;x_1,y_1)\right]\Phi(x_1,y_1). \quad (24)\end{aligned}$$

By comparing Eqs.(23),(24), we conclude that the Poisson kernel must satisfy the following identity:

$$K_{H_2}(x_3,y_3;x_1,y_1) = \int_{-\infty}^{\infty} dx_2\ K_{H_2}(x_3,y_3;x_2,y_2)\cdot K_{H_2}(x_2,y_2;x_1,y_1). \quad (25)$$

In this form, it is tempting to identify the y coordinate as a "time" variable, then the convolution formula looks just like an evolution identity.

2.2 Proof of the convolution formula on H_2

One can prove the convolution formula, Eq.(25), using the trick of residue theorem. To achieve this purpose, it is convenient to formulate the following lemma (the proof is a straightforward computation).

Lemma 2.1.

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{d\xi}{(\xi-\alpha)(\xi-\bar\alpha)(\xi-\beta)(\xi-\bar\beta)} \\ &= \left(\frac{2\pi i}{\alpha-\beta}\right)\left[\frac{1}{(\alpha-\bar\alpha)(\alpha-\bar\beta)}+\frac{1}{(\beta-\bar\beta)(\bar\alpha-\beta)}\right]. \quad (26)\end{aligned}$$

Here we assume $\alpha,\beta\in\mathbb{C}$, and $Im(\alpha)$, $Im(\beta)>0$, $\bar\alpha,\bar\beta$ are complex conjugates of α and β, respectively.

Now we introduce a convenient notation for simplifying the expressions,

$$x_{21} := x_2 - x_1 = -x_{12}, \qquad y_{21} := y_2 - y_1 = -y_{12}. \quad (27)$$

After substituting these notations into the definition of the Poisson kernel, Eq.(5), we obtain an explicit form of the convolution formula, Eq.(25),

Theorem 2.2.

$$\begin{aligned} K_{H_2}(x_3, y_3; x_1, y_1) &= \int_{-\infty}^{\infty} dx_2 \; K_{H_2}(x_3, y_3; x_2, y_2) \cdot K_{H_2}(x_2, y_2; x_1, y_1) \\ \Rightarrow \quad \frac{y_{31}}{x_{31}^2 + y_{31}^2} &= \frac{1}{\pi} \int_{-\infty}^{\infty} dx_2 \; \frac{y_{32} \cdot y_{21}}{(x_{32}^2 + y_{32}^2)\,(x_{21}^2 + y_{21}^2)}. \end{aligned} \tag{28}$$

Here we assume the "time-ordering", $y_1 < y_2 < y_3$.

Proof. In order to apply lemma 2.1. to evaluate the integration over x_2 in Eq.(28), we need to identify the poles of the integrand:

$$x_{32}^2 + y_{32}^2 = 0 \Rightarrow x_2 = x_3 \pm i y_{32}, \qquad x_{21}^2 + y_{21}^2 = 0 \Rightarrow x_2 = x_1 \pm i y_{21}. \tag{29}$$

According to the assignment of time-ordering, $y_1 < y_2 < y_3$, we identify the parameters in Lemma 2.1 as follows.

$$\alpha := x_3 + i y_{32} \quad \beta := x_1 + i y_{21}. \tag{30}$$

The relevant combinations are

$$\alpha - \bar{\alpha} = 2 i y_{32}, \qquad \beta - \bar{\beta} = 2 i y_{21}, \tag{31}$$

$$\alpha - \beta = x_{31} + i(y_{32} - y_{21}), \qquad \alpha - \bar{\beta} = x_{31} + i y_{31}. \tag{32}$$

Putting all ingredients together, we have

$$\frac{1}{(\alpha - \bar{\alpha})(\alpha - \bar{\beta})} + \frac{1}{(\beta - \bar{\beta})(\bar{\alpha} - \beta)} = \left(\frac{1}{2i}\right)\left(\frac{y_{31}}{x_{31}^2 + y_{31}^2}\right)\left[\frac{x_{31} + i(y_{32} - y_{21})}{y_{32} y_{21}}\right], \tag{33}$$

and setting $\xi = x_2$, we have

$$\begin{aligned} &\frac{1}{\pi} \int_{-\infty}^{\infty} dx_2 \frac{y_{32} \cdot y_{21}}{(x_{32}^2 + y_{32}^2)\,(x_{21}^2 + y_{21}^2)} \\ &= \frac{y_{32} \cdot y_{21}}{\pi} \int_{-\infty}^{\infty} \frac{d\xi}{(\xi - \alpha)(\xi - \bar{\alpha})(\xi - \beta)(\xi - \bar{\beta})}. \end{aligned} \tag{34}$$

Then by Lemma 2.1. one obtains Eq.(28). □

3 Poisson Kernel Associated with a Disk Domain

3.1 Formulating the convolution formula on D_2

In the case of a disk domain, the unique solution of the Laplace's equation,

$$\Delta\Psi = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)\Psi(\rho,\phi) = 0, \tag{35}$$

subject to the boundary condition, $\Psi(R,\chi) = f(\chi)$, can be expressed in terms of the Poisson integral formula,

$$\Psi(\rho,\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\chi\ K_{D_2}(\rho,\phi;R,\chi)f(\chi), \quad 0 \le \rho \le R, \tag{36}$$

where the Poisson kernel $K_{D_2}(\rho,\phi;R,\chi)$ is defined as

$$K_{D_2}(\rho,\phi;R,\chi) := \frac{R^2-\rho^2}{R^2+\rho^2-2R\rho\cos(\chi-\phi)}. \tag{37}$$

In order to illustrate the convolution property of the Poisson kernel, it is helpful to recast the Poisson intergral formula, Eq.(36), as

$$\Psi(2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_1\ K_{D_2}(2,1)\Psi(1), \tag{38}$$

where we have made a short-hand notation by using a single number to represent the polar coordinates of a point on the disk, that is,

$$1 := (\rho_1,\phi_1),\ 2 := (\rho_2,\phi_2),\ 3 := (\rho_3,\phi_3). \tag{39}$$

By composing two Poisson integral formulas,

$$\begin{aligned}\Psi(3) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_2\ K_{D_2}(3,2)\Psi(2) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_2\ K_{D_2}(3,2)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_1\ K_{D_2}(2,1)\Psi(1)\right],\end{aligned} \tag{40}$$

and the fact that the boundary-radius independence of the Poisson integral formula,

$$\Psi(3) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_1\ K_{D_2}(3,1)\Psi(1), \tag{41}$$

we then deduce that the Poisson kernel must satisfy the following convolution formula

$$K_{D_2}(3,1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_2\ K_{D_2}(3,2)K_{D_2}(2,1). \tag{42}$$

3.2 Proof of the convolution formula on D_2

One way to evaluate the convolution formula, Eq.(42), is to employ the contour integral technique in the complex analysis, and the residue theorem reduces the computations into algebraic manipulations.

Firstly, we combine the polar coordinates (ρ_k, ϕ_k) into complex variables:

$$z_k := \rho_k e^{i\phi_k}, \quad k = 1,2,3, \quad \text{assuming that } \rho_3 < \rho_2 < \rho_1. \tag{43}$$

One then show that Poisson kernel, Eq.(9), associated with a disk boundary becomes

$$\begin{aligned} K_{D_2}(2,1) &= \frac{\rho_1^2 - \rho_2^2}{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\phi_2 - \phi_1)} \\ &= \frac{z_1\bar{z}_1 - z_2\bar{z}_2}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} = \frac{(\rho_1^2 - \rho_2^2)(z_1 z_2)}{(z_1 - z_2)(\rho_1^2 z_2 - \rho_2^2 z_1)}. \end{aligned} \tag{44}$$

Substituting the complex forms of the Poisson kernels to the convolution formula, Eq.(42), and changing the integration variables, $d\phi_2 = \dfrac{dz_2}{iz_2}$, I transform the convolution formula into the following contour integral,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi_2\ K_{D_2}(3;2)K_{D_2}(2;1) \\ = &\frac{(\rho_1^2 - \rho_2^2)(\rho_2^2 - \rho_3^2)}{2\pi i}\oint_{|z_2|=\rho_2} \frac{dz_2}{z_2(z_2 - z_1)(\bar{z}_2 - \bar{z}_1)(z_2 - z_3)(\bar{z}_2 - \bar{z}_3)} \\ = &\frac{(\rho^2)_{12}\cdot(\rho^2)_{23}}{2\pi i\cdot \bar{z}_1\bar{z}_3}\oint_{|z_2|=\rho_2} \frac{z_2 dz_2}{(z_2 - z_1)\left(z_2 - \dfrac{\rho_2^2}{\bar{z}_1}\right)(z_2 - z_3)\left(z_2 - \dfrac{\rho_2^2}{\bar{z}_3}\right)}. \end{aligned} \tag{45}$$

Out of the four simple poles of the integrand, only two of them locate inside the contour $|z_2| = \rho_2$, which are $z_2 = z_3$ and $z_2 = \dfrac{\rho_2^2}{\bar{z}_1} = \dfrac{\rho_2^2}{\rho_1^2}z_1$. By the residue theorem we conclude that the contour integral is equal to

$$\frac{(\rho^2)_{12}\cdot(\rho^2)_{23}}{\bar{z}_1\bar{z}_3}\left[\mathrm{Res}(z_2 = z_3) + \mathrm{Res}\left(z_2 = \frac{\rho_2^2}{\rho_1^2}z_1\right)\right]. \tag{46}$$

The evaluations of the residue is a straightforward computation. In the end, we verify that the final result of Eq.(42) is equal to

$$\frac{\rho_1^2-\rho_3^2}{(z_1-z_3)(\bar{z}_1-\bar{z}_3)}=K_{D_2}(3;1). \tag{47}$$

4 Conformal Mapping between two Poisson Kernels

4.1 Conformal transformations between two domains

In this section, I show that the Poisson kernels Eqs.(5),(9), together with their generalizations, associated with different boundary-value problems of the Laplace's equation can be related via a conformal transformation.

In order to simplify the notations, it is convenient to introduce two sets of complex variables, $z := x+iy \in H_2$ and $\omega := \rho \cdot e^{i\phi} \in D_2$ and the conformal transformation mapping H_2 to D_2 is

$$\omega = f(z) = \left(\frac{i-z}{i+z}\right)R \quad \Longleftrightarrow \quad z = f^{-1}(\omega) = i\left(\frac{R-\omega}{R+\omega}\right). \tag{48}$$

Based on this definition, one can check the following facts:

- The conformal transformation of special points:

$$z=0 \Longleftrightarrow \omega=R, \quad z=\pm 1 \Longleftrightarrow \omega=\pm iR, \quad z=\pm\infty \Longleftrightarrow \omega=-R, \tag{49}$$

$$\omega=0 \Longleftrightarrow z=i, \quad \omega=i\rho \Longleftrightarrow z=\frac{\rho+iR}{i\rho+R}. \tag{50}$$

- The conformal transformation maps boundary to boundary,

$$\partial H_2=\{y=0\} \quad \Longleftrightarrow \quad \partial D_2=S^1=\{\rho=R\}, \tag{51}$$

$$e^{i\phi}=\frac{i-x}{i+x} \quad \Longleftrightarrow \quad x=\tan(\phi/2). \tag{52}$$

- The concentric circles, $\rho := |\omega|$ =const., in D_2 are mapped to a family of "floating bubbles" in H_2, whose centers lie at $(0,\sigma)$, with radii equal to s,

$$r^2 := \left(\frac{\rho}{R}\right)^2 = \frac{(i-z)(-i-\bar{z})}{(i+z)(-i+\bar{z})} \Rightarrow x^2+(y-\sigma)^2=s^2. \tag{53}$$

Here we define the following symbols:

$$\sigma := \frac{1+r^2}{1-r^2}=\frac{R^2+\rho^2}{R^2-\rho^2}, \quad s := \frac{2r}{1-r^2}=\frac{2R\rho}{R^2-\rho^2}, \quad \sigma^2-s^2=1. \tag{54}$$

One can parametrize a floating bubble, Eq.(53), by an angle θ:

$$x(\theta) = s\cdot\sin(\theta), \quad y(\theta) = \sigma - s\cdot\cos(\theta) \;\Rightarrow\; x(\theta)^2 + y(\theta)^2 = 2\sigma y(\theta) - 1. \quad (55)$$

The angle of the concentric circles, ϕ, Eq.(52), in D_2 is related to the angle of the floating bubble, θ, Eq.(55), in H_2,

$$e^{i\phi} = \frac{e^{i\theta} - r}{1 - re^{i\theta}} \quad \Rightarrow \quad \frac{d\phi}{d\theta} = \frac{1 - r^2}{(1 - re^{i\theta})(1 - re^{-i\theta})} = 1/y(\theta). \qquad (56)$$

- The y =const. parallel lines in H_2 are mapped to a family of "shrinking balloons" inside D_2, whose centers lie at $(\tau, 0)$, with radii equal to t,

$$\omega =: u + iv = \frac{(1 - x^2 - y^2) + 2ix}{x^2 + (y+1)^2} \quad \Rightarrow \; (u - \tau)^2 + v^2 = t^2. \qquad (57)$$

Here we define the following symbols:

$$\tau := -\left(\frac{y}{y+1}\right)\cdot R, \quad t := \left(\frac{1}{y+1}\right)\cdot R, \quad \tau^2 - t^2 = \left(\frac{y-1}{y+1}\right)\cdot R^2 = -(1+2\tau)R^2. \qquad (58)$$

One can parametrize a "shrinking balloon" by an angle φ:

$$u(\varphi) = \tau + t\cdot\cos(\varphi), \quad v(\varphi) = t\cdot\sin(\varphi) \;\Rightarrow\; u(\varphi)^2 + v(\varphi)^2 = 2\tau u(\varphi) + (1+2\tau)R^2. \qquad (59)$$

The x coordinates of the $y =$ const. parallel lines in H_2, is related to the angle of the shrinking balloon, φ in D_2,

$$\frac{u}{R} + 1 = \frac{2(y+1)}{x^2 + (y+1)^2} = \left(\frac{\tau}{R} + 1\right) + \left(\frac{t}{R}\right)\cos(\varphi) \quad \Rightarrow \quad \frac{dx}{d\varphi} = \frac{R}{u + R}. \qquad (60)$$

4.2 Conformal transformation of the Poisson kernels

4.2.1 Conformal transformation of the disk Poisson kernels

In this subsection, we study the conformal transformations of the Poisson kernels. To begin with, I compute the conformal transformation of the disk

Poisson kernel, Eq.(9),

$$K_{D_2}(\omega_2 = f(z_2),\ \omega_1 = f(z_1)) = \frac{\omega_1\bar{\omega}_1 - \omega_2\bar{\omega}_2}{(\omega_1-\omega_2)(\bar{\omega}_1-\bar{\omega}_2)} = \frac{|f(z_1)|^2 - |f(z_2)|^2}{|f(z_1)-f(z_2)|^2}$$
$$= \frac{y_2(z_1\bar{z}_1+1) - y_1(z_2\bar{z}_2+1)}{|z_1-z_2|^2}. \qquad (61)$$

It is clear that the result is not equal to the UHF Poisson kernel on H_2,

$$K_{D_2}(\omega_2 = f(z_2),\ \omega_1 = f(z_1)) \neq K_{H_2}(z_2, z_1) = \frac{\mathrm{Im}(z_2-z_1)}{|z_1-z_2|^2}. \qquad (62)$$

For this reason, we need to define a more general disk Poisson kernel on the complex plane, whose domain is a disk centered at ζ, with radius $|\xi_1 - \zeta| = \rho_1$, $\Omega := \{z|\ |z-\zeta| \leq \rho_1\}$,

$$\mathcal{K}_\Omega(\xi_2, \xi_1; \zeta) := \frac{|\xi_1-\zeta|^2 - |\xi_2-\zeta|^2}{|\xi_1-\xi_2|^2}, \quad \text{where} \quad \xi_1 \in \partial\Omega, \quad \xi_2 \in int(\Omega). \qquad (63)$$

Recalling that the floating bubble image of the conformal transformation, Eq.(55), one can now verify the following statement:

Theorem 4.1.

$$K_{D_2}(\omega_2 = f(z_2),\ \omega_1 = f(z_1))\ d\phi_1 = \mathcal{K}_\Omega(z_2, z_1; \zeta = i\sigma_1)\ d\theta_1. \qquad (64)$$

Proof. Recall the Cartesian equation of the "floating bubble", Eqs.(53),(55), we have

$$K_{D_2}(\omega_2 = f(z_2),\ \omega_1 = f(z_1)) = \frac{y_2(2\sigma_1 y_1) - y_1(2\sigma_2 y_2)}{|z_1-z_2|^2}. \qquad (65)$$

On the other hand, the expression of the general disk Poisson kernel is

$$\mathcal{K}_\Omega(z_2, z_1;\ \zeta = i\sigma_1) = \frac{|z_1 - i\sigma_1|^2 - |z_2 - i\sigma_1|^2}{|z_1-z_2|^2} \qquad (66)$$
$$= \frac{|z_1|^2 - |z_2|^2 + 2\sigma_1(y_2-y_1)}{|z_1-z_2|^2} = \frac{2(\sigma_1-\sigma_2)y_2}{|z_1-z_2|^2}. \qquad (67)$$

Finally, incorporating the relation between two measures of the contour integrals, Eq.(56),

$$d\phi_1 = (1/y_1)\ d\theta_1 \qquad (68)$$

we show that the two sides of Eq.(64) are equal. □

4.2.2 Conformal transformation of the UHP Poisson kernels

Similar analysis also applies to the conformal transformation of the UHP Poisson kernel, where we have,

$$K_{H_2}(z_2 = f^{-1}(\omega_2),\ z_1 = f^{-1}(\omega_1)) = \frac{1}{\pi}\ \frac{\mathrm{Im}[\ f^{-1}(\omega_1) - f^{-1}(\omega_2)\ ]}{|f^{-1}(\omega_1) - f^{-1}(\omega_2)|^2}$$
$$= \frac{1}{2\pi R\ |\omega_1 - \omega_2|^2}\left[(u_2 + R)(u_1^2 + v_1^2) - (u_1 + R)(u_2^2 + v_2^2) + R^2(u_1 - u_2)\right], \quad (69)$$

and one observes that

$$\pi K_{H_2}(z_2 = f^{-1}(\omega_2),\ z_1 = f^{-1}(\omega_1)) \neq K_{D_2}(\omega_1,\ \omega_2) = \frac{(u_1^2 + v_1^2) - (u_2^2 + v_2^2)}{|\omega_1 - \omega_2|^2}. \quad (70)$$

The correct correspondence, again, is given in terms of the general disk Poisson kernel associated with the "shrinking balloon", weighted by the proper measure, Eq.(60),

Theorem 4.2. *The conformal image of the UHF Poisson kernel can be expressed as a general disk Poisson kernel.*

$$K_{H_2}(z_2 = f^{-1}(\omega_2),\ z_1 = f^{-1}(\omega_1))\ dx_1 = \frac{1}{2\pi}\ \mathcal{K}_\Omega(\omega_2, \omega_1; \zeta = \tau_1)\ d\varphi_1. \quad (71)$$

Proof. The general disk Poisson kernel associated with the "shrinking balloon", Eq.(57), is

$$\mathcal{K}_\Omega(\omega_2, \omega_1; \zeta = \tau_1) = \frac{|\omega_1 - \tau_1|^2 - |\omega_2 - \tau_1|^2}{|\omega_1 - \omega_2|^2} = \frac{(u_1^2 + v_1^2) - (u_2^2 + v_2^2) - 2\tau_1(u_1 - u_2)}{|\omega_1 - \omega_2|^2}. \quad (72)$$

Incorporating the relation between two measures of the contour integrals, Eq.(60),

$$dx_1 = \frac{R}{u_1 + R}\ d\varphi_1 \quad (73)$$

we show that the numerators of two sides of Eq.(71) are equal.

$$\text{L.H.S.}\quad (u_2 + R)(u_1^2 + v_1^2) - (u_1 + R)(u_2^2 + v_2^2) + R^2(u_1 - u_2) \quad (74)$$
$$= \text{R.H.S.}\quad \left[(u_1^2 + v_1^2) - (u_2^2 + v_2^2) - 2\tau_1(u_1 - u_2)\right](u_1 + R). \quad (75)$$

□

4.3 General convolution formula for disk Poisson kernels

It is a general feature that the Möbius transformation, e.g., Eq.(48), maps a circle to a circle (It is natural to identify a straight line as a circle with an infinite radius). In this subsection, I study a more general convolution formula associated with the disk Poisson kernels based on these correspondences between level curves on different domains.

We start with observing the common features of Eqs.(53),(57). In either case, we have a sequence of "inclusive circles", where the circle of small radius is completely contained inside the circle of larger radius. The crucial difference, as compared with the discussions in Sec.2, and Sec.3, is that these circles may not be concentric. However, if we think more carefully, it is not necessary that the regions (being a half plane in H_2 or a disk in D_2) to have paralle boundaries, what is relevant for the convolution formula is the inclusion of domains, and the shape of the boundaries do not matter. Consequently, we can now conjecture a more general convolution formula for the disk Poisson kernels:

Theorem 4.3. *The general disk Poisson kernels, $\mathcal{K}_{\Omega_m}$, associated with disks Ω_m, satisfy the following integral identity,*

$$\mathcal{K}_{\Omega_1}(z_3, z_1; \zeta_1) = \int_{\partial\Omega_2} d\phi_2 \; \mathcal{K}_{\Omega_2}(z_3, z_2(\phi_2); \zeta_2) \cdot \mathcal{K}_{\Omega_1}(z_2(\phi_2), z_1; \zeta_1)|_{z_2(s)\in\partial\Omega_2}. \tag{76}$$

Here the disks Ω_m with center at ζ_m and radius ρ_m is defined as

$$\Omega_m := \{z_m | \; z_m \in \mathbb{C}, \; |z_m - \zeta_m| \leq \rho_m\}, \tag{77}$$

and they obey the inclusion rule (the boundaries of different disk do not intersect)

$$\Omega_m \subsetneq \Omega_n, \; if \; \rho_m < \rho_n. \tag{78}$$

Note that: (1) the discussions of Eqs.(25),(42) in Sec.2 and Sec.3, now become special cases of this more general formula. (2) the integral identity implies that the R.H.S. of Eq.(76) is independent of the location of the center ζ_2.

4.4 Proof of the convolution formula for general disk Poisson kernels

To simplify the calculations, I define the following complex variables (inclusion of domains: $\Omega_2 \subsetneq \Omega_1$),

$$z_1 \in \partial\Omega_1, z_1 \notin \Omega_2, \qquad X := z_1 - \zeta_1 := \rho_1 e^{i\phi_1}, \qquad \bar{z}_1 = \bar{\zeta}_1 + \frac{\rho_1^2}{z_1 - \zeta_1}, \tag{79}$$

$$z_2 \in \partial\Omega_2, z_2 \in \Omega_1, \qquad z := z_2 - \zeta_2 := \rho_2 e^{i\phi_2}, \qquad \bar{z}_2 = \bar{\zeta}_2 + \frac{\rho_2^2}{z}, \tag{80}$$

$$z_3 \in int(\Omega_2) \subset int(\Omega_1), \qquad \xi_3 := z_3 - \zeta_2 := \rho_3 e^{i\phi_3}, \qquad \bar{z}_3 = \bar{\zeta}_2 + \frac{\rho_3^2}{z_3 - \zeta_2}. \tag{81}$$

The general disk Poisson kernels can now be expressed in terms of the complex notations:

$$\mathcal{K}_{\Omega_1}(2,1) = \frac{|z_1 - \zeta_1|^2 - |z_2 - \zeta_1|^2}{|z_1 - z_2|^2} = \frac{\rho_1^2 z - (z_2 - \zeta_1)(\bar{\zeta}_{21} z + \rho_2^2)}{(z_2 - z_1)[(\bar{\zeta}_2 - \bar{z}_1)z + \rho_2^2]}, \tag{82}$$

$$\mathcal{K}_{\Omega_2}(3,2) = \frac{|z_2 - \zeta_2|^2 - |z_3 - \zeta_2|^2}{|z_2 - z_3|^2} = \frac{(\rho_2^2 - \rho_3^2)z}{(z - \xi_3)[(\bar{\zeta}_2 - \bar{z}_3)z + \rho_2^2]}. \tag{83}$$

Together with integration measure, $d\phi_2 = \dfrac{dz}{iz}$, the R.H.S. of the convolution formula, Eq.(76), becomes,

$$\frac{1}{2\pi}\oint_{\partial\Omega_2} d\phi_2\, \mathcal{K}_{\Omega_2}(z_3, z_2; \zeta_2) \cdot \mathcal{K}_{\Omega_1}(z_2, z_1; \zeta_1)|_{z_2 \in \partial\Omega_2} = \frac{\rho_2^2 - \rho_3^2}{2\pi i}\oint_{\partial\Omega_2} dz\, \frac{\text{(I)}}{\text{(II)}}, \tag{84}$$

where $\quad \text{(I)} := \rho_1^2 z - (z_2 - \zeta_1)[\bar{\zeta}_{21} z + \rho_2^2], \tag{85}$

and $\quad \text{(II)} := (z_2 - z_1)[(\bar{\zeta}_2 - \bar{z}_1)z + \rho_2^2](z_2 - z_3)[(\bar{\zeta}_2 - \bar{z}_3)z + \rho_2^2]. \tag{86}$

The following new complex variables are useful to simplify the expression of the integral,

$$z := z_2 - \zeta_2, \qquad \xi_1 := z_1 - \zeta_2, \qquad \xi_3 := z_3 - \zeta_2. \tag{87}$$

$$\text{(I)} \;:=\; \rho_1^2 z - (z + \zeta_{21})(\bar{\zeta}_{21} z + \rho_2^2), \tag{88}$$

$$\text{(II)} \;:=\; (z - \xi_1)(\bar{\xi}_1 z - \rho_2^2)(z - \xi_3)(\bar{\xi}_3 z - \rho_2^2). \tag{89}$$

Next, we need to identify the simple poles of the integrand, and it should be clear that only the 2nd and the 3rd terms in (II) (the denominator of the integrand) are relevant for the residue theorem. It takes some skills to simplify the respective residues, and the details are provided in the App.A. Summing both residues, Eqs.(94),(97), we obtain the final result of convolution integral,

$$(\rho_2^2-\rho_3^2)\cdot\left[\text{Res.}\left(z=\frac{\rho_2^2}{\bar{\xi}_1}\right)+\text{Res.}(z=\xi_3)\right]=\frac{|z_1-\zeta_1|^2-|z_3-\zeta_1|^2}{|z_1-z_3|^2}=\mathcal{K}_{\Omega_1}(z_3,z_1;\zeta_1). \tag{90}$$

5 Summary and discussion

The main point of this paper is to formulate the convolution property of the Poisson kernels of the Laplace's equation in the UHP and disk domains. Explicit mathematical identities of these convolution formulas, Eq.(25),(42), are proved via the residue theorem in the complex analysis. The interconnections between these two boundary-value problems are clarified via a conformal transformation, Eq.(48). This correspondence also motivates a more general convolution formula for disk Poisson kernels, Eq.(76).

At the pedagogical level, this work shows that how geometric/physical arguments can lead to non-trivial mathematical identities, and it also illustrates the power of contour integral in dealing with complicated integrations.

There are several obvious generalizations of this work, which I hope to make progress on. These include: (1) higher-dimensional generalizations of the convolution formulas for the Poisson kernel of Laplace's equations, at least in the case of domains with special symmetry, (2) A general proof of these mathematical identities for general domains. This may require the use of Schwarz–Christoffel transformation and/or other mathematical techniques.

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6 Reference

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A Computations of the residues for the disk Poisson kernel

A.1 Computations of the first residue

To simplify the residue of the pole of the convolution integral at $z = \dfrac{\rho_2^2}{\bar{\xi}_1}$,

$$\text{Res}\left[z = \frac{\rho_2^2}{\bar{\xi}_1}\right] = \frac{\left[\rho_1^2\bar{\xi}_1 - \left(\rho_2^2 + \bar{\xi}_1\zeta_{21}\right)\left(\bar{\xi}_1 + \bar{\zeta}_{21}\right)\right](\xi_3 - \xi_1)}{\left(\rho_2^2 - \bar{\xi}_1\xi_3\right)\left(\rho_2^2 - |\xi_1|^2\right)|\xi_1 - \xi_3|^2}, \tag{91}$$

we introduce the following notations

$$W := \rho_2^2 - \bar{\xi}_1\xi_3, \quad X := \xi_1 + \zeta_{21} = z_1 - \zeta_1, \quad \text{and} \quad Y := \xi_3 + \zeta_{21} = z_3 - \zeta_1. \tag{92}$$

The numerator of Eq.(91) is

$$\begin{aligned}
& \rho_1^2\bar{\xi}_1(\xi_3 - \xi_1) - \left[\rho_2^2 + \bar{\xi}_1(Y - \xi_3)\right]\bar{X}(\xi_3 - \xi_1) \\
= \; & -\rho_1^2|\xi_1|^2 + \rho_1^2\rho_2^2 - \rho_1^2\rho_2^2 + \rho_1^2\bar{\xi}_1\xi_3 \\
& -\rho_2^2\bar{X}(Y - X) + \bar{\xi}_1\xi_3\bar{X}(Y - X) - \bar{\xi}_1 Y\bar{X}(\xi_3 - \xi_1) \\
= \; & \rho_1^2\left(\rho_2^2 - |\xi_1|^2\right) - \rho_1^2 W - \rho_2^2\bar{X}Y + \rho_2^2|X|^2 \\
& -\bar{\xi}_1\xi_3|X|^2 + |\xi_1|^2\bar{X}Y \\
= \; & \left(\rho_1^2 - \bar{X}Y\right)\left(\rho_2^2 - |\xi_1|^2\right) - \left(\rho_1^2 - |X|^2\right)W = \left(\rho_1^2 - \bar{X}Y\right)\left(\rho_2^2 - |\xi_1|^2\right),
\end{aligned} \tag{93}$$

then the residue, Eq.(91), is equal to

$$\frac{\rho_1^2 - \bar{X}Y}{W \cdot |\xi_1 - \xi_3|^2}. \tag{94}$$

A.2 Computations of the second residue

To simplify the residue of the pole of the convolution integral at $z = \xi_3$,

$$\text{Res}[z = \xi_3] = \frac{\left[\rho_1^2\xi_3 - \left(\rho_2^2 + \xi_3\bar{\zeta}_{21}\right)(\xi_3 + \zeta_{21})\right]\left(\bar{\xi}_3 - \bar{\xi}_1\right)}{\left(\rho_2^2 - \bar{\xi}_1\xi_3\right)\left(\rho_2^2 - |\xi_3|^2\right)|\xi_1 - \xi_3|^2}, \tag{95}$$

we recall the following notations,

$$W := \rho_2^2 - \bar{\xi}_1\xi_3, \quad X := \xi_1 + \zeta_{21} = z_1 - \zeta_1, \quad \text{and} \quad Y := \xi_3 + \zeta_{21} = z_3 - \zeta_1,$$

then the numerator of Eq.(95) becomes

$$\begin{aligned}
& \rho_1^2\xi_3(\bar{\xi}_3 - \bar{\xi}_1) - \left[\rho_2^2 + \xi_3(\bar{X} - \bar{\xi}_1)\right] Y(\bar{\xi}_3 - \bar{\xi}_1) \\
= \;& \rho_1^2|\xi_3|^2 - \rho_1^2\rho_2^2 + \rho_1^2\rho_2^2 - \rho_1^2\bar{\xi}_1\xi_3 \\
& -\rho_2^2 Y(\bar{Y} - \bar{X}) + \bar{\xi}_1\xi_3 Y(\bar{Y} - \bar{X}) - \xi_3\bar{X}Y(\bar{\xi}_3 - \bar{\xi}_1) \\
= \;& -\rho_1^2\left(\rho_2^2 - |\xi_3|^2\right) + \rho_1^2 W \\
& -\rho_2^2|Y|^2 + \rho_2^2\bar{X}Y + \bar{\xi}_1\xi_3|Y|^2 - |\xi_3|^2\bar{X}Y \\
= \;& \left(-\rho_1^2 + \bar{X}Y\right)\left(\rho_2^2 - |\xi_3|^2\right) + \left(\rho_1^2 - |Y|^2\right) W. \qquad (96)
\end{aligned}$$

Consequently, the second residue, Eq.(95) is equal to

$$\frac{-\rho_1^2 + \bar{X}Y}{W \cdot |\xi_1 - \xi_3|^2} + \frac{\rho_1^2 - |Y|^2}{(\rho_2^2 - |\xi_3|^2)|\xi_1 - \xi_3|^2}. \qquad (97)$$

Poisson 核之摺積性質

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中文摘要:

由二維拉普拉斯方程式之精確解實例出發，本文探討不同區域Poisson核之摺 積公式。透過複變分析中的留數定理，可以直接證明這些數學等式。最後，我 透過複平面保角變換，聯繫不同區域之Poisson核及其摺積公式。

Abstract:

Based on examples of the exact solutions of the two-dimensional Laplace's equation, I derive the convolution formulas among the Poisson kernels associated with given domains. These mathematical identities can be verified directly using the residue theorem in the complex analysis. The interconnections of convolution formulas for different domains are established via a conformal transformation.

Keywords:

Laplace's equation, Poisson kernel, convolution formula, contour integral, residue theorem.