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Toda Fields and Logarithmic Conformal Field Theory

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Abstract

We discuss the logarithmic operators in Toda field theory.

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1 Introduction

Two dimensional conformal field theories(CFT) have been the subject of intense interest since the publication of the seminal paper of Belavin, Polyakov and Zamolodichikov.[1] Their work form the basis of string theory and have many important applications to statistical mechanics and condensed matter physics. Despite being extraordinary comprehensive, the possibility that correlation functions may contain logarithms was not considered at all in that paper. In 1993 Gurarie [2] introduced the logarithmic operators preserving conformal invariance that produce the logarithms in the correlation functions. These logarithms have been found in the models such as the Wess-Zumino-Witten (WZW) model on the supergroup $GL(1,1)$[3], non-minimal $c_{p,q}$ models [2,4,5], the WZW models at level 0 [6,7,8] and many others. However, the logarithmic conformal field theories remain difficult to work with, and are not yet classified in the similar way as the standard CFT.

Logarithmic operators are a straightforward generalization of the primary operators.[9,10,11] The so-called primary states created by the primary operators are annihilated by all the Virasoro generators L_n with $n > 0$ and are eigenstates of the of the Virasoro generator L_0,

$$L_0|A\rangle = h|A\rangle \tag{1}$$

Logarithmic operators are a generalization of that to non-diagonalizable matrices. The logarithmic states are also annihilated by L_n with $n > 0$, but form a Jordan block with respect to L_0

$$\begin{aligned} L_0|C\rangle &= h|C\rangle && (2) \\ L_0|D\rangle &= h|D\rangle + |C\rangle, && (3) \end{aligned}$$

or equivalently

$$L_0 \begin{pmatrix} |C\rangle \\ |D\rangle \end{pmatrix} = \begin{pmatrix} h & 0 \\ 1 & h \end{pmatrix} \begin{pmatrix} |C\rangle \\ |D\rangle \end{pmatrix}. \tag{4}$$

L_0 can be interpreted as the Hamiltonian in standard CFT. But it can be seen in (2), (3) that L_0 is not hermitian since $L_0^\dagger \neq L_0$. Therefore, the logarithmic states cannot be unitary.

Let us review how logarithmic operators appear in logarithmic CFT and, firstly, calculate two-point correlation functions by conformal invariance. Under an infinitesimal conformal transformation, $z \to z + \epsilon(z)$, a primary operator transforms as [12]

$$\delta C(z) = \epsilon(z)\frac{\partial C(z)}{\partial z} + h\frac{\partial \epsilon(z)}{\partial z}C(z), \tag{5}$$

where h is the anomalous dimension. Its logarithmic partner $D(z)$ transforms in a different way

$$\delta D(z) = \epsilon(z)\frac{\partial D(z)}{\partial z} + \frac{\partial \epsilon(z)}{\partial z}(hD(z) + C(z)). \tag{6}$$

The two-point correlation function of a primary operator $A(z)$ is invariant under the translation with $\epsilon(z) = \epsilon =$ constant, dilatation with $\epsilon(z) = \epsilon z$ and special conformal transformation (SCT) with $\epsilon(z) = \epsilon z^2$, fix the only correlation function up to a constant

$$\langle A(z)A(w)\rangle = \frac{b}{(z-w)^{2h}}, \tag{7}$$

where b is an arbitrary constant. This correlation function will satisfy the differential equations,

$$\begin{aligned}
&(\partial_z + \partial_w)\langle A(z)A(w)\rangle = 0,\\
&(\partial_z + \partial_w + 2h)\langle A(z)A(w)\rangle = 0,\\
&(w^2\partial_z + w^2\partial_{z_2} + 2h(z+w))\langle A(z)A(w)\rangle = 0.
\end{aligned} \tag{8}$$

As for the correlation function $\langle C(z)C(w)\rangle$, it will satisfy the same set of equations (8) as $\langle A(z)A(w)\rangle$. The correlation functions involving $D(z)$ should satisfy slightly different equations,

$$\left\{\begin{array}{c}
(\partial_z + \partial_w)\langle D(z)C(w)\rangle = 0,\\
(\partial_z + \partial_w + 2h)\langle D(z)C(w)\rangle + \langle C(z)C(w)\rangle = 0,\\
(z^2\partial_z + w^2\partial_w + 2h(z+w))\langle D(z)C(w)\rangle + 2z\langle C(z)C(w)\rangle = 0,
\end{array}\right. . \tag{9}$$

$$\left\{\begin{array}{c}
(\partial_z + \partial_w)\langle C(z)D(w)\rangle = 0,\\
(\partial_z + \partial_w + 2h)\langle C(z)D(w)\rangle + \langle C(z)C(w)\rangle = 0,\\
(z^2\partial_z + w^2\partial_w + 2h(z+w))\langle C(z)D(w)\rangle + 2w\langle C(z)C(w)\rangle = 0,
\end{array}\right. . \tag{10}$$

$$\left\{\begin{array}{c}
(\partial_z + \partial_w)\langle D(z)D(w)\rangle = 0,\\
(\partial_z + \partial_w + 2h)\langle D(z)D(w)\rangle + \langle C(z)D(w)\rangle + \langle D(z)C(w)\rangle = 0,\\
(z^2\partial_z + w^2\partial_w + 2h(z+w))\langle D(z)D(w)\rangle + 2z\langle C(z)D(w)\rangle + 2w\langle D(z)C(w)\rangle = 0,
\end{array}\right. . \tag{11}$$

The only solution of the above equations is

$$\begin{aligned}
\langle C(z)C(w)\rangle &= 0,\\
\langle D(z)C(w)\rangle &= \langle C(z)D(w)\rangle = \frac{b}{(z-w)^{2h}},\\
\langle D(z)D(w)\rangle &= -2b\frac{ln(z-w)}{(z-w)^{2h}}.
\end{aligned} \tag{12}$$

It can be seen that the operator $D(z)$ justifies its name of a logarithmic operator, as its two- point correlation function contains a logarithm.

In this paper we plan to discuss the logarithmic operators in the Toda field theory, which can be taken as a generalization of the Liouville conformal field theory. So let us review the Liouville conformal field theory first.

2 Logarithmic Operators in Liouville Conformal Field Theory

The Liouville conformal field theory was first introduced by Polyakov [13], aiming to calculate the path integral measure coming from the interaction between closed bosonic strings. The action arising from the calculation is given by

$$S = \frac{1}{4\pi} \int d^2x \sqrt{g} \, [g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + QR\phi + 4\pi\mu e^{2b\phi}]. \tag{13}$$

The first term of the integrand represents the kinetic free scalar field, the second term is the curvature coupling term and the third term is the Liouville exponential potential term. Q is the coupling parameter, R is the Ricci scalar curvature, and μ is the scale parameter. As the Gauss-Bonnet theorem is used, one can regard the coupling of the curvature with the field as adding a background charge of $-Q$ at infinite. The central charge of the action (13) is [14]

$$c = 1 + 6Q^2 \tag{14}$$

and the energy-momentum stress tensor is

$$T(z) = Q\partial^2\phi - \partial\phi\partial\phi, \tag{15}$$

where the complex coordinates z can be written in terms of our two-dimensional Euclidean space coordinates as $z = x_1 + ix_2$, $\partial = \partial/\partial z$. Though adding the Liouville exponential potential term, the action (13) remain conformal. This is true if and only if

$$Q = b + \frac{1}{b}. \tag{16}$$

All the important parameters for the Liouville CFT remain the same after adding the Liouville exponential. In this theory, the primary fields are vertex operators of the form

$$V_\alpha =: e^{2\alpha\phi(z)} : \tag{17}$$

with the conformal dimensions

$$\Delta_\alpha = \alpha(Q - \alpha). \tag{18}$$

The primary fields have the usual operator product expansion with the stress tensor

$$T(z)V_\alpha(0) = \frac{\Delta_\alpha}{z^2} V_\alpha(0) + \frac{1}{z}\partial V_\alpha(0) + ... \tag{19}$$

For every value of α, there are two operator with the same dimension Δ_α given by (18), V_α and $V_{Q-\alpha}$. When

$$\alpha = \frac{Q}{2}, \tag{20}$$

there are also two primary operators with the same dimension. The second one is

$$\frac{\partial}{\partial\alpha}V_\alpha\,|_{\alpha=\frac{Q}{2}} = 2\phi(z)e^{Q\phi(z)} \tag{21}$$

This is called the puncture operator in the Liouville CFT. We might expect $\frac{\partial}{\partial\alpha}V_\alpha\,|_{\alpha=\frac{Q}{2}}$ is a logarithmic operator, but this is not true. To see this clearly, we differentiate Eq.(19) with respect to α, which gives

$$T(z)\left(\frac{\partial}{\partial\alpha}V_\alpha(0)\right) = \frac{\partial\Delta_\alpha}{\partial\alpha}\frac{1}{z^2}V_\alpha(0) + \frac{\Delta_\alpha}{z^2}\left(\frac{\partial}{\partial\alpha}V_\alpha(0)\right) + \frac{1}{z}\partial\left(\frac{\partial}{\partial\alpha}V_\alpha(0)\right) + ... \tag{22}$$

The first term of the right-hand side of Eq.(22) vanishes, since

$$\frac{\partial\Delta_\alpha}{\partial\alpha}|_{\alpha=\frac{Q}{2}} = Q - 2\alpha = 0. \tag{23}$$

There is one way to obtain correctly the second primary operator, which is the logarithmic operator, when $\alpha = Q/2$. Instead of Eq.(22), one find

$$T(z)D_\alpha(0) = \frac{\Delta_\alpha}{z^2}D_\alpha(0) + \frac{1}{z^2}V_\alpha(0) + \frac{1}{z}\partial D_\alpha(0) + ... \tag{24}$$

and the logarithmic operator D_α can be written as

$$D_\alpha = \left(\frac{\partial\Delta_\alpha}{\partial\alpha}\right)^{-1}\frac{\partial}{\partial\alpha}V_\alpha. \tag{25}$$

Now let us proceed to calculate the two-point correlation functions. Under an infinitesimal conformal transformation, $z \to z + \epsilon(z)$

$$\delta V_\alpha(z) = \epsilon(z)\,\partial V_\alpha(z) + \Delta_\alpha\,\partial\epsilon(z)V_\alpha(z), \tag{26}$$

$$\delta D_\alpha(z) = \epsilon(z)\partial D_\alpha(z) + \partial\epsilon(z)(\Delta_\alpha D_\alpha(z) + V_\alpha(z)). \tag{27}$$

We can obtain the similar correlation function

$$\langle D_\alpha(z)D_{Q-\alpha}(w)\rangle = -2b\frac{ln(z-w)}{(z-w)^{2\Delta_\alpha}}. \tag{28}$$

involving a logarithm.

3 Logarithmic Operators in Conformal Toda Field Theory

Conformal Toda field theory is much more complicated than the Liouville field theory. This theory provided an important example of CFT with high spin symmetry. The Lagrangian of the $\mathfrak{sl}(n)$ conformal Toda field theory [15] has the form

$$\mathcal{L} = \frac{1}{4\pi}\left(\partial_\alpha\phi\right)^2 + \mu\sum_{k=1}^{n-1}e^{2b(e_k,\phi)} \tag{29}$$

where ϕ is the two-dimensional $(n-1)$-component scalar field that $\phi = (\phi_1 ... \phi_{n-1})$, b is the dimensionless coupling constant and (e_k, ϕ) denotes the scalar product, in which vectors e_k are the simple roots of the Lie algebra $\mathfrak{sl}(n)$ with the Cartan matrix of the scalar products $K_{ij} = (e_i, e_j)$

$$K_{ij} = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -1 & 0 \\ 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{30}$$

Total normalization of the Lagrangian given in Eq.(29) is chosen in such a way that

$$\phi_i(z, \bar{z})\phi_j(0,0) = -\delta_{ij}\, ln|z|^2 + ... \quad \text{at } z \to 0. \tag{31}$$

The action of the Toda field theory on a surface with metric $g_{\mu\nu}$ is

$$S_{Toda} = \frac{1}{4\pi} \int d^2x \sqrt{g}\, [g^{\alpha\beta}(\partial_\alpha \phi, \partial_\beta \phi) + R(Q, \phi) + 4\pi\mu \sum_{k=1}^{n-1} e^{b(e_k, \phi)}], \tag{32}$$

where R is the scalar curvature of the background metric. If the background charge Q is related with the parameter b as

$$Q = \left(b + \frac{1}{b}\right)\frac{\rho}{2} \tag{33}$$

with ρ being a Weyl vector, half of the sum of all positive roots, then the theory based on Eq.(32) is conformally invariant. The central charge of the action (32) is

$$c = n - 1 + 3Q^2 = (n-1)(1 + n(n+1)(b + b^{-1})^2) \tag{34}$$

and the energy-momentum stress tensor is

$$T(z) = (Q, \partial^2 \phi) - (\partial \phi)^2, \tag{35}$$

The primary operators of this theory are vertex operators parameterized by a $(n-1)$-component vector parameter α

$$V_\alpha = e^{2(\alpha, \phi)} \tag{36}$$

with the conformal dimensions

$$\Delta_\alpha = (\alpha, Q - \alpha). \tag{37}$$

The primary fields have the usual operator product expansion with the stress tensor

$$T(z)V_\alpha(0) = \frac{\Delta_\alpha}{z^2} V_\alpha(0) + \frac{1}{z}\partial V_\alpha(0) + ... \tag{38}$$

Now, we know that the logarithmic operator D_α can be written as

$$D_\alpha = \left(\left(\frac{\partial \Delta_\alpha}{\partial \alpha} \right)^{-1}, \frac{\partial}{\partial \alpha} V_\alpha \right). \tag{39}$$

and

$$T(z) D_\alpha(0) = \frac{\Delta_\alpha}{z^2} D_\alpha(0) + \frac{1}{z^2} V_\alpha(0) + \frac{1}{z} \partial D_\alpha(0) + ... \tag{40}$$

With the same procedure we also obtain the correlation function

$$\langle D_\alpha(z) D_{Q-\alpha}(0) \rangle = -2b \frac{lnz}{z^{2\Delta_\alpha}}. \tag{41}$$

which contains a logarithm.

4 Discussions

As shown in Eq.(39), we find that the logarithmic operators exist in the conformal Toda field theory. Although it is similar to that of Liouville theory, but rarely referred in the literature. Conformal Toda field theory is notorious for its complexity and high spin W_n symmetry structures, nevertheless, this theory may be regarded as an arena for finding some new properties about the logarithmic CFT.

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戶田場和對數共型場論

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摘 要

我們討論在戶田共型場論中的對數算子。

關鍵字: 戶田共型場論、對數共型場論