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The Riemann Zeros and the Spectrum of a Dirac Fermion on $3+1$ Rindler Spacetimes

Der-Pyng Maa, * Yeunjyr Ju, † Shyang-Ling Lou ‡

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Abstract

We propose a spectral realization of the Riemann zeros as the energy levels of a massive Dirac fermion on $3+1$ Rindler spacetimes.

*Deparment of Economics, Tunghai University, Taichung, Taiwan
†Deparment of Computer Science and Information Engineering, Tunghai University, Taiwan
‡Deparment of Physics, Tunghai University, Taichung, Taiwan

1 Introduction

The Riemann hypothesis proposed by Bernhard Riemann (1859) is a conjecture that the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

has nontrivial complex zeros with real part 1/2, which is important for its connection with the distribution of prime numbers. Some mathematicians consider the hypothesis as one of the most important unsolved problem in pure mathematics. One interesting and prominent idea related to physics goes back to Pólya and Hilbert, where they suggested that perhaps the imaginary part ε_n of the Riemann zeros $s_n = \frac{1}{2} + i\varepsilon_n$ are the eigenvalues of a quantum mechanical Hamiltonian[1]. This idea maybe nearly empty, unless there exist some Hamiltonian that can intrinsically explain why the real part of the Riemann zeros is 1/2.

In 1999 Berry, Keating and Connes [2][3][4] suggested that a spectral realization of the Riemann zeros could be achieved by quantizing the classical Hamiltonian $H_{BK} = xp$, where x and p are the position and momentum of a particle moving in one dimension. The connection between xp and Riemann zeros relies on the regularization schemes. Berry and Keating regularized the model by introducing the Planck cell in phase space with sides l_x and l_p and area $l_x l_p = 2\pi\hbar$, such that $|x| > l_x$ and $|p| > l_p$. The number of states $N_{BK}(\varepsilon)$, with an energy below ε ($\varepsilon \gg 1$) is given semiclassically by

$$N_{BK}(\varepsilon) = \frac{1}{h} \int_{l_x}^{\varepsilon/l_p} dx \int_{l_p}^{\varepsilon/x} dp. \tag{2}$$

The result is

$$N_{BK}(\varepsilon) = \frac{\varepsilon}{h}(ln\frac{\varepsilon}{h} - 1) + 1. \tag{3}$$

Taking into account a Maslov phase $-1/8$,[2] Berry and Keating obtain the formula

$$N_{BK}(\varepsilon) = \frac{\varepsilon}{2\pi}(ln\frac{\varepsilon}{2\pi} - 1) + \frac{7}{8}. \tag{4}$$

where the energy is measured in units $\hbar = 1$. Surprisingly, Eq.(4) coincides asymtotically with the smooth part of the Riemann formula [5],

$$N_R(\varepsilon) = \frac{\varepsilon}{2\pi}(ln\frac{\varepsilon}{2\pi} - 1) + \frac{7}{8} + o(1) + N_{osc}(\varepsilon), \tag{5}$$

in the critical strip, i.e. $0 < Re(s) < 1, 0 < Im(s) < \varepsilon$, where the oscillatory part of $N_R(\varepsilon)$ is given by

$$N_{osc}(\varepsilon) = \frac{1}{\pi}\, Im\, \{ln\, \zeta(\frac{1}{2} + i\varepsilon)\} = o(ln\varepsilon) \tag{6}$$

To quantize the xp model, Berry and Keating used the Hamiltonian [2]

$$\hat{H}_{BK} = \frac{1}{2}(x\hat{p} + \hat{p}x) = -i(x\frac{d}{dx} + \frac{1}{2}), \tag{7}$$

where x belongs to the real line and $\hat{p} = -id/dx$ is the momentum operator. If x is restricted to the positive half-line, then Eq.(7) is equivalent to

$$\hat{H}_{BK} = \sqrt{x}\hat{p}\sqrt{x}. \tag{8}$$

The eigenfunctions, with eigenvalue ε, are given by

$$\psi_\varepsilon(x) = \frac{1}{\sqrt{2\pi}}\, x^{-\frac{1}{2}+i\varepsilon}, \tag{9}$$

where the normalization constant is obtained by using the Dirac's delta function. There is something interesting in Eq.(9) that deserves to be noticed. If x and p under the scale transformation, i.e. $x \to Kx$, $p \to K^{-1}p$, with $K > 0$. If the transformation is taken using $K = n$, where n is an integer, then

$$\psi_\varepsilon(x) \to \sum_{n=1}^{\infty} \psi_\varepsilon(nx) = \frac{1}{\sqrt{2\pi}}\, x^{-\frac{1}{2}+i\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}-i\varepsilon}} = \frac{1}{\sqrt{2\pi}}\, x^{-\frac{1}{2}+i\varepsilon}\zeta(\frac{1}{2} - i\varepsilon). \tag{10}$$

One gets the Riemann zeros ε_n as Eq.10 vanishes.

The Berry-Keating xp model was revisited in 2011 in terms of classical Hamiltonian $H = x(p+1/p)$ whose quantizations contain the smooth approximation of the Riemann zeros. [6][7] Then, using the generalized reformulation of the Hamiltonian $H = U(x)p + V(x)/p)$,[8][9] the Hamiltonian $H = x(p + 1/p)$ was shown to be equivalent to the massive Dirac equation on $1+1$ Rindler spacetime, that is the natural arena to study accelerated observers and the Hawking radiation in black holes and the corresponding temperature.

2 Dirac Fermion on $1 + 1$ Rindler Spacetime

The $1+1$ dimensional Minkowski spacetime is defined by a pair of coordinates (x^0, x^1), and the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{11}$$

where $\eta_{00} = -1$, $\eta_{11} = 1$, $\eta_{01} = \eta_{10} = 0$. We use the units of the speed of light $c = 1$. Applying the coordinate transformation to the Minkowski spacetime

$$x^0 = z sinh(t), \;\; x^1 = z cosh(t), \tag{12}$$

the induced metric becomes

$$ds^2 = dz^2 - z^2 dt^2, \tag{13}$$

where z and t are the Rindler space and time coordinates respectively[10][11] Let us consider an observer whose world line is given by (12), with $z = l > 0$, which is a constant value. The proper time τ measured by the observer is defined by $ds^2 = -d\tau^2$.

The relation to t follows from (13) is $\tau = lt$.The observer's trajectory (12), written in terms of its proper time, reads

$$x^0 = lsinh(\tau/l), \ \ x^1 = lcosh(\tau/l), \tag{14}$$

Define a constant proper acceleration a as the Minkowski norm of the vector

$$a^\mu = \frac{d^2x^\mu}{d\tau^2}, \quad a^\mu a_\mu = -a^2, \quad a = \frac{1}{l}, \tag{15}$$

In this approach to a spectral realization of the Riemann zeros, the observer is chosen to have an acceleration $a = 1/l$. To study the dynamics of a Dirac fermion, let use a representation of the Dirac's gamma matrices in the $1+1$ dimensions,[12][13][14]

$$\gamma^0 = \sigma^1, \quad \gamma^1 = -i\sigma^2, \quad \gamma^5 \equiv \gamma^0\gamma^1 = \sigma^3, \tag{16}$$

where $\sigma^i(i = 1, 2, 3)$ are the Pauli matrices. γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad \mu, \nu = \ 0, 1. \tag{17}$$

The Dirac fermion ψ is a two component spinor

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger\gamma^0 = (\psi_+^\dagger, \psi_-^\dagger), \tag{18}$$

where $\psi_\pm^\dagger$ are the hermitian conjugate of $\psi_\pm$. γ^5 defines the chirality of the fermions. The fields $\psi_\pm$ are the chiral components of ψ, $\gamma^5\psi_\pm = \mp\psi_\pm$. Now introducing the light cone $x^\pm$ and its derivatives $\partial_\pm = \partial/\partial x^\pm$

$$\begin{aligned} x^\pm &= x^0 \pm x^1 = \pm z e^{\pm t}, &(19)\\ \partial_\pm &= \frac{1}{2}(\partial_0 \pm \partial_1) = \pm\frac{1}{2}e^{\mp t}(\partial_z \pm \frac{1}{z}\partial_t). &(20)\end{aligned}$$

Under a Lorentz transformation with boost parameter ζ, i.e. the velocity $v = tanh\zeta$, the light cone coordinates, its derivatives and the Dirac spinors transform as

$$x^\pm \to e^{\mp\zeta}x^\pm, \quad \partial_\pm \to e^{\pm\zeta}\partial_\pm, \quad \psi_\pm \to e^{\pm\zeta/2}\psi_\pm, \tag{21}$$

and the Rindler coordinates transform as

$$t \to t - \zeta, \quad \ z \to z. \tag{22}$$

Hence if we define the spinor

$$\chi_\pm \equiv e^{\pm t/2}\ \psi_\pm, \tag{23}$$

then $\chi_\pm$ remain invariant under the Lorentz transformation.

The action of a Dirac fermion with mass m is

$$\begin{aligned} S &= \int_V d^2x\bar{\psi}(\gamma^\mu\partial_\mu + im)\psi && (24)\\ &= \int_V dx^+dx^-\{\psi_-^\dagger\partial_+\psi_- + \psi_+^\dagger\partial_-\psi_+ \frac{im}{2}(\psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_-)\} && (25)\\ &= \int_{-\infty}^{\infty} dt\int_l^\infty dz\{\chi_-^\dagger(\partial_t + z\partial_z + \frac{1}{2})\chi_- + \chi_+^\dagger(\partial_t - z\partial_z - \frac{1}{2})\chi_+ \\ &\quad +imz(\chi_-^\dagger\chi_+ + \chi_+^\dagger\chi_-)\}. && (26)\end{aligned}$$

V has a boundary ∂V corresponding to the worldline $z = l > 0$. The action principle applied to the expression of (26) gives the Dirac equation

$$(\partial_t \pm z\partial_z \pm \frac{1}{2})\chi_\mp + imz\chi_\pm = 0, \tag{27}$$

and the boundary condition

$$\chi_-^\dagger(l,t)\delta\chi_-(l,t) = \chi_+^\dagger(l,t)\delta\chi_+(l,t) \tag{28}$$

for any t. The equation (27) can be rewritten as the Rindler Hamiltonian H_R,

$$i\partial_t\chi = H_R\chi, \quad \chi = \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix}, \tag{29}$$

$$H_R = \begin{pmatrix} -i(z\partial_z + \frac{1}{2}) & mz \\ mz & i(z\partial_z + \frac{1}{2}) \end{pmatrix} = \sqrt{z}\hat{p}_z\sqrt{z}\sigma_z + mz\sigma_x, \tag{30}$$

where $\hat{p}_z = -i\partial/\partial z$, is the momentum operator associated to the radial coordinate z. The operator

$$\sqrt{z}\hat{p}_z\sqrt{z} = \frac{1}{2}(z\hat{p}_z + \hat{p}_z z) = -i(z\partial z + \frac{1}{2}) \tag{31}$$

coincides with the quantization of the classical xp Hamiltonian, $\hat{H}_{BK}$, proposed by Berry and Keating, where x is the radial Rindler coordinate. Thus H_R consists of two copies of $\hat{H}_{BK}$, with different signs corresponding to opposite chiralities that are coupled by the mass term. Besides of these, since the Hamiltonian H_R is hermitian and the boundary condition (28), χ_+ and χ_- have the relation [13]

$$\chi_+ = -ie^{i\theta}\chi_- \tag{32}$$

at the boundary $z = l$. The quantity $-ie^{i\theta}$ has the physical meaning of the phase shift produced by the reflection of the fermion with the boundary.

The eigenvalues and the eigenvectors of the Hamiltonian H_R, are given by the solutions of the Schödinger equation

$$H_R\,\chi = E\,\chi \quad \chi_\pm(z,t) = e^{-iEt}f_\pm(z), \quad z \geq l \tag{33}$$

that satisfy the boundary condition (32). The equations for $f_\pm(z)$ that follows from Eq.(30) are

$$(z\partial_z + \frac{1}{2} \pm iE)f_\pm \mp imzf_\pm = 0. \tag{34}$$

These two coupled equations will lead to the second order differential equations

$$[z^2\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial z} - (\frac{1}{2} \pm iE)^2 - m^2z^2]f_\pm = 0, \tag{35}$$

whose general solution is a linear combination of the modified Bessel functions

$$f_\pm(z) = c_1 e^{\mp i\pi/4}K_{\frac{1}{2}\pm iE}(mz) \pm c_2 e^{\mp i\pi/4}I_{\frac{1}{2}\pm iE}(mz). \tag{36}$$

The phase $e^{\mp i\pi/4}$ follows from (34). If we take $E = i/2$ for convenience, the second term of the right-hand side of Eq.(36) diverges exponentially as $z \to \infty$, ; this will force the constant $c_2 = 0$. Setting $c_1 = 1$, one has

$$f_\pm(z) = e^{\mp i\pi/4}K_{\frac{1}{2}\pm iE}(mz), \tag{37}$$

which yields the eigenfunctions

$$\chi_\pm(z,t) = e^{-iEt\mp i\pi/4}K_{\frac{1}{2}\pm iE}(mz), \tag{38}$$

Plugging (38) into (32) yields the equation for the eigenenergies

$$-e^{i\theta}K_{\frac{1}{2}-iE}(ml) + K_{\frac{1}{2}+iE}(ml) = 0. \tag{39}$$

Choosing $\theta = \pi$ and using the asymptotic behavior of modified Bessel function

$$K_{\frac{1}{2}+iE}(ml) \to \sqrt{\frac{\pi}{2E}}(\frac{2E}{ml})^{1/2}e^{-\pi E/2}(\frac{2E}{mle})^{iE}, \tag{40}$$

for $E \gg 1$, one gets

$$K_{\frac{1}{2}+iE}(ml) + K_{\frac{1}{2}-iE}(ml) = 0 \to cos[E(ln\frac{2E}{mle})] = 0. \tag{41}$$

The number of eigenvalues $N(E)$ in the interval $[0, E]$ is given in the asymptotic limit $E \gg ml$ by

$$N(E) \simeq \frac{E}{\pi}(ln\frac{2E}{ml} - 1) - \frac{1}{2} \tag{42}$$

This expression agrees asymtotically with the smooth part of the Riemann formula (5), with the identifications

$$E = \frac{\varepsilon}{2}, \quad ml = 2\pi. \tag{43}$$

with which (42) becomes

$$N(\varepsilon) \simeq \frac{\varepsilon}{2\pi}(ln\frac{\varepsilon}{2\pi} - 1) - \frac{1}{2}. \tag{44}$$

However, the constant term 7/8 in Eq.(5) is not reproduced in this case.

3 Dirac Fermion on $3+1$ Rindler Spacetime

The $3+1$ dimensional Minkowski spacetime is defined by the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{45}$$

where $x^\mu = (x^0, x^1, x^2, x^3)$ and the metric with signature $(-1, 1, 1, 1)$. Applying the coordinate transformation to the Minkowski spacetime

$$x^0 = zsinh(t), \quad x^3 = zcosh(t), \quad x^1 = x \quad x^2 = y, \tag{46}$$

the metric becomes the Rindler's metric

$$ds^2 = dx^2 + dy^2 + dz^2 - z^2 dt^2. \tag{47}$$

The Dirac equation on this spacetime is

$$\gamma^\mu \nabla_\mu \psi + im\psi = 0, \tag{48}$$

where the spinor covariant derivative

$$\nabla^\mu \psi = \partial^\mu \psi + \frac{1}{4} \omega^\mu_{bc} \gamma^b \gamma^c \psi, \quad \omega^\mu_{bc} = -\omega^\mu_{cb}. \tag{49}$$

We introduce the orthonormal frame with the vierbein $e^a{}_\mu$, then

$$e^a{}_\mu \omega^\mu_{bc} = \omega^a_{bc}, \;\; e^a{}_\mu \nabla^\mu \psi = \nabla^a \psi. \tag{50}$$

The orthonormal basis is

$$e^1 = dx, \quad e^2 = dy, \quad e^3 = dz, \quad e^0 = zdt, \tag{51}$$

$$e^a = e^a{}_\mu dx^\mu. \tag{52}$$

The spin connection coefficients ω^a_{bc} are introduced by

$$de^a + \omega^a_{bc}\, e^b \wedge e^c = 0. \tag{53}$$

The nonzero connection coefficients are

$$\omega^0_{03} = -\omega^0_{30} = \frac{1}{z}. \tag{54}$$

Defining the spin connection one-form $\omega^a{}_b$ by the first Cartan's structure equation

$$de^a + \omega^a{}_b \wedge e^b = 0, \tag{55}$$

the only nonzero one-form connection is

$$\omega^0{}_3 = \frac{1}{z}\, e^0. \tag{56}$$

Defining the curvature 2-form by the second Cartan's structure equation

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \tag{57}$$

the only component of the curvature 2-form is

$$R^0{}_3 = d\omega^0{}_3 = -\frac{1}{z^2}e^3 \wedge e^0 + \frac{1}{z^2}e^3 \wedge e^0 = 0. \tag{58}$$

The Rindler spacetime is locally flat. The Dirac equation (48) now becomes

$$\begin{aligned} &\gamma^1\partial_x\psi + \gamma^2\partial_y\psi + \gamma^3[\partial_z\psi + \frac{1}{2z}\psi] \\ &+\frac{1}{z}\gamma^0\partial_t\psi + im\psi = 0 \end{aligned} \tag{59}$$

Choosing the Dirac representation $\gamma^i = \beta\alpha^i$, $\gamma^0 = \beta$, with

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3. \tag{60}$$

To find the solutions of (59), the four-spinor is decomposed into two smaller two-spinor

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{61}$$

then (59) splits into the Pauli-like equations

$$\sigma \cdot \partial\chi + \sigma^3\frac{\chi}{2\mathrm{z}} = -\frac{1}{z}\partial_t\phi + m\phi, \tag{62}$$

$$\sigma \cdot \partial\phi + \sigma^3\frac{\phi}{2\mathrm{z}} = -\frac{1}{z}\partial_t\chi - m\chi. \tag{63}$$

where $\sigma \cdot \partial = \sigma^1\partial_{\mathrm{x}} + \sigma^2\partial_{\mathrm{y}} + \sigma^3\partial_{\mathrm{z}}$. One can find the positive energy stationary states

$$\phi(x) = \Phi(\mathbf{x})\mathrm{e}^{-\mathrm{iEt}}, \quad \chi(\mathrm{x}) = \Psi(\mathbf{x})\mathrm{e}^{-\mathrm{iEt}}, \tag{64}$$

and (62) and (63) turns into

$$\sigma \cdot \partial\Phi + \sigma^3\frac{\Phi}{2\mathrm{z}} = i\frac{E}{z}\Psi - m\Psi, \tag{65}$$

$$\sigma \cdot \partial\Psi + \sigma^3\frac{\Psi}{2\mathrm{z}} = i\frac{E}{z}\Phi + m\Phi. \tag{66}$$

Substituting

$$\Phi = z^{-1/2}f(\mathbf{x}), \quad \Psi = \mathrm{z}^{-1/2}\mathrm{g}(\mathbf{x}), \tag{67}$$

into (65) and (66), we have the form

$$\triangle f + [\frac{E^2}{z^2} + m^2]f = -i\frac{E}{z^2}\sigma^3 g, \tag{68}$$

$$\triangle g + [\frac{E^2}{z^2} + m^2]g = -i\frac{E}{z^2}\sigma^3 f, \tag{69}$$

where $\triangle = \partial_x^2 + \partial_y^2 + \partial_z^2$. Taking the sum $F = f + g$ and the difference $G = f - g$, (68) and (69) decouple into

$$\triangle F + i\frac{E}{z^2}\sigma^3 F + [\frac{E^2}{z^2} + m^2]F = 0, \tag{70}$$

$$\triangle G - i\frac{E}{z^2}\sigma^3 G + [\frac{E^2}{z^2} + m^2]G = 0. \tag{71}$$

Consider the planar momentum states and the 2-spinor structure

$$F = \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} e^{i(k_x x + k_y y)}, \quad G = \begin{pmatrix} G_+(z) \\ G_-(z) \end{pmatrix} e^{i(k_x x + k_y y)}, \tag{72}$$

and thus

$$f = \frac{1}{2}\begin{pmatrix} F_+ + G_+ \\ F_- + G_- \end{pmatrix} e^{i(k_x x + k_y y)}, \quad g = \frac{1}{2}\begin{pmatrix} F_+ - G_+ \\ F_- - G_- \end{pmatrix} e^{i(k_x x + k_y y)}, \tag{73}$$

one ends up with the following equations

$$\frac{d^2 F_\pm}{dz^2} + [-\lambda^2 + \frac{\omega(\omega \pm i)}{z^2}]F_\pm = 0, \tag{74}$$

$$\frac{d^2 G_\pm}{dz^2} + [-\lambda^2 + \frac{\omega(\omega \mp i)}{z^2}]G_\pm = 0, \tag{75}$$

where $\lambda^2 = k_x^2 + k_y^2 - m^2$. The solutions can be expressed in terms of the modified Bessel functions

$$F_+(z) = \sqrt{z} K_{\frac{1}{2} - iE}(\lambda z), \tag{76}$$

$$F_-(z) = \sqrt{z} K_{\frac{1}{2} + iE}(\lambda z). \tag{77}$$

From (74) and (75), it can be seen that the pairs F_+, G_- and F_-, G_+ fulfill the same equation respectively. The functions must satisfy the proportion relations

$$G_+ = c_+ F_-, \quad G_- = c_- F_+. \tag{78}$$

The proportional constants c_+ and c_- can be derived from (68), (69), and (73),

$$c_+ = \pm i e^{-i\theta}, \quad c_- = \mp i e^{i\theta}. \tag{79}$$

The overall 4-spinor solution is

$$\psi = \frac{e^{-iET}}{\sqrt{z}} \frac{N}{2} \begin{pmatrix} F_+(z) + ie^{-i\theta}F_-(z) \\ F_-(z) - ie^{i\theta}F_+(z) \\ F_+(z) - ie^{-i\theta}F_-(z) \\ F_-(z) + ie^{i\theta}F_+(z) \end{pmatrix} e^{i(k_x x + k_y y)}, \tag{80}$$

where N is the normalization constant. When ψ vanishes at $z = l$ and choosing $\theta = \frac{\pi}{2}$, we can obtain the equation

$$K_{\frac{1}{2}-iE}(\lambda l) + K_{\frac{1}{2}+iE}(\lambda l) = 0. \tag{81}$$

Now with

$$E = \frac{\varepsilon}{2}, \quad \lambda l = 2\pi, \tag{82}$$

then the number of eigenvalues $N(E)$ in the interval $[0, E]$ given in the asymptotic limit $E \gg \lambda l$ is

$$N(E) \simeq \frac{\varepsilon}{2\pi}(ln\frac{\varepsilon}{2\pi} - 1) \tag{83}$$

This expression also agrees asymtotically with the smooth part of the Riemann formula (5).

4 Discussions

It is very interesting to find that the massive Dirac fermion on $3+1$ Rindler spacetime, not only in the $1+1$ dimensional case, is also related to the Riemann zeros problem. This is by no means the final story. There must be a deeper origin for the relationship of the spectrum of the Dirac theory and the Riemann zeros. The discussions about the oscillatory part $N_{osc}(\varepsilon)$ of the Riemann formula by the Dirac theory are rare. It will be interesting to explore whether there exists a generalized Dirac theory that relates to the zeros of the L-functions, Dirichlet L-functions, or related to the generalized Riemann hypothesis.

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黎曼零點和四維林德勒時空狄拉克費米子能階的關係

馬德平　　　朱允執　　　婁祥麟

摘 要

我們討論在四維林德勒時空中，具質量狄拉克費米子的能階和黎曼零點分佈之間的關係。

關鍵字: 黎曼零點、狄拉克費米子