Tunghai Science Vol. 1: 49−63
July, 1999

An Asymptotic Formula for the Solution of a Singular Perturbation Problem

Shing-Liang Lu*

Abstract

We study, in the rectangle $\Omega = (0,a) \times (0,b)$, the Dirichlet boundary value problem for the elliptic partial differential equation

$$Lu \equiv -\varepsilon\Delta u + pu_x + gu_y + qu = f,$$

where $0 < \varepsilon \ll 1$, Δ is the Laplacian operator, and the functions p, g, q and f satisfy certain hypotheses; in particular, $p > 0$, $q \geq 0$. We construct a formal asymptotic expansion of the solution u of this problem for small ε. This expansion contains the solution of the reduced equation and boundary layer functions. The parabolic boundary layer functions satisfy a parabolic equation with an unbounded coefficient. We transform the parabolic equation into a heat equation to develop properties of the parabolic boundary layer. Estimates for the remainder in the expansion are established that are of the order of magnitude of powers of ε.

Keywords: singular perturbation, boundary layer.

1 Introduction

We study, in the rectangle $\Omega = (0,a) \times (0,b)$, the Dirichlet boundary value problem for an elliptic partial differential equation of the form

$$Lu \equiv -\varepsilon\Delta u + pu_x + gu_y + qu = f \tag{1.1}$$

with boundary conditions

$$u(x,y) = 0 \quad \text{on} \quad \Gamma, \tag{1.1a}$$

where ε is a small parameter $0 < \varepsilon \ll 1$, Δ is the Laplacian operator, and Γ is the boundary of Ω. The function u satisfies the differential equation in Ω and the boundary conditions on Γ. We shall

*Department of Mathematics, Tunghai University, Taichung 407, TAIWAN

impose the following hypotheses:

$(C1)$ f is a smooth function in $\overline{\Omega}$;
$(C2)$ q is a smooth and nonnegative function in $\overline{\Omega}$;
$(C3)$ g is a smooth function in $\overline{\Omega}$ with $g(x,0)=0$, and $g(x,b)=0$;
$(C4)$ p is a smooth and positive function in $\overline{\Omega}$.

In hypotheses $(C1)$-$(C4)$, "smooth" means at least "C^2". From the assumption that q is nonnegative in $\overline{\Omega}$, it follows that the problem $(1.1),(1.1a)$ has a unique solution $u(x,y,\varepsilon)$. Under the conditions assumed, it is well known that for a fixed value of ε, the solution $u(x,y,\varepsilon)$ is smooth in $\overline{\Omega}$.

Confronted with a singular perturbation problem, one is usually interested in constructing an asymptotic approximation of the function u. Such an approximation contains so called "boundary layer" term, which are asymptotically equivalent to zero everywhere in Ω exception for a small neighborhood of a part of the boundary Γ. Moreover, the boundary layer functions occur singularities near the corners $(0,0)$, $(0,b)$.

Some work on problems related to (1.1), $(1.1a,b)$ are contained in Levinson [6], Eckhaus-De Jaeger [1], Shih-Kellogg [5], Vasil'eva, Butuzov, and Kalachev [7], and A. M. Il'in [3]. In [6] there is derived an asymptotic approximation to the solution with an error that is uniformly $O(\sqrt{\varepsilon})$ in subregions that are bounded away from characteristic boundaries. The Levinson approximation contains a boundary layer term, but does not contain terms representing the parabolic layer that is present near the characteristic boundary of the problem. The papers of [1] and [5] study the equation

$$-\varepsilon\Delta u+u_x+u=f$$

in the unit square. (The problem considered in [5] is somewhat more general.) In this case, the subcharacteristic curves of the reduced equation are lines parallel to the x-axis. An asymptotic expansion is constructed which has an error that is uniformly small in the square. The expansion contains both boundary layer terms and parabolic layer terms, the latter having importance on the horizontal sides, $y=0$ and $y=1$, of the square. In [7, p.93], the term u_x is replaced by $\varepsilon^{\alpha}u_x$, where $\alpha<\frac{1}{2}$, and the resulting asymptotic expansion is studied. In [3, p.121], the term u_x is replaced by $a(x,y)u_x$ with $a(x,y)>0$.

In the present work, we study the equation (1.1). Our purpose is to construct a formal asymptotic expansion of the solution u of (1.1) for small ε and give a proof of its uniform validity in the closed rectangle. This expansion contains the solution of the reduced equation and boundary layer functions whose roles are to correct the discrepancy between the boundary data and the boundary values of this reduced problem. The parabolic boundary layer functions satisfy a parabolic equation with an unbounded coefficient. We transform the parabolic equation into a heat equation to develop properties of the parabolic boundary layer. In fact, the equation (1.1) can be transformed into an equation whose characteristic curves are parallel to the new “x-axis” by making a change of coordinates. The speciality of this paper is the easy computation of the approximate solution and a convenient estimate of the error term ϕ under Euclidean coordinates. The main tool used in this paper for estimating the solution of the elliptic boundary value problem $(1.1),(1.1a)$ is furnished by the so-call maximum principle and the concept of barrier function. For the proof of the maximum principle see Eckhaus and De Jager [1]. We are ready to state the maximum principle.

MAXIMUM PRINCIPLE. *Let Φ and Ψ be twice continuously differentiable functions in Ω such that*

$$|L[\Phi]| \leq L[\Psi] \quad in\ \Omega,$$

$$|\Phi| \leq \Psi \quad on\ \Gamma.$$

Then

$$|\Phi| \leq \Psi \quad in\ \overline{\Omega}.$$

In Chapter 2, there are given some properties of the solution of the reduced equation; that is, the equation that is obtained from (1.1) by setting $\varepsilon = 0$. Chapter 3 contains the full boundary layer analysis. Chapter 4 gives bound for the error in the expansion.

2 The Reduced Equation

In order to obtain a first rough approximation of the function $u(x,y,\varepsilon)$ for small values of the parameter ε, we consider a function $v_0(x,y)$ which satisfies the differential eqation obtained from (1.1) by putting $\varepsilon = 0$. This hyperbolic equation is called the reduced equation and reads as follows:

$$pv_{0x} + gv_{0y} + qv_0 = f.$$

The function $v_0(x,y)$ can satisfy only one of the prescribed boundary conditions

$$v_0(0,y) = 0,$$

and

$$v_0(a,y) = 0.$$

Theorem 2.1 *There exists a positive constant C independent of* ε *such that the inequality*

$$|u(x,y,\varepsilon)| \leq Cx \tag{2.1}$$

holds uniformly in the closure of Ω *for all values of* ε*.*

Proof: We introduce the barrier function $\Psi(x) = Cx$, where C is some positive constant independent of ε. By taking C sufficiently large it follows that the inequalities

$$|u(x,y,\varepsilon)| \leq \Psi(x)$$

on the boundary of Ω and

$$|L[u]| \leq L[\Psi]$$

in Ω can be satisfied for all values of ε. Applying the maximum principle, we get the desired inequality (2.1) uniformly valid in the closure of Ω for all values of ε. □

According to Theorem 2.1, as ε tends to zero, we are led to the inequality

$$|v_0(x,y)| \leq Cx.$$

Therefore $v_0(0,y) = 0$ is the proper condition for the solution $v_0(x,y)$. The solution of this reduced equation is an ingredient in the asymptotic expansion of u. The characteristic curves of this reduced equation, also called subcharacteristic curves of equation (1.1) are defined by $y = \phi(x,y_0)$, where the function ϕ defined on $0 \leq x \leq a$ for all $0 \leq y_0 \leq b$, with $\phi(x,0) = 0$, $\phi(x,b) = b$ satisfies the initial value problem

$$\frac{d\phi}{dx} = \frac{g(x,\phi)}{p(x,\phi)}, \tag{2.2}$$

$$\phi(0,y_0) = y_0.$$

By existence and uniqeness theorem and the properties of maximum solution of ordinary differential equations, the characteristic curves cover the whole rectangle Ω as y_0 varies in the interval $[0,b]$ under the assumptions for p and g. Also, by the uniqueness theorem for solutions of ordinary differential equations, $\phi(x,y_0)$ is strictly increasing in y_0 so we can write $y_0 = \psi(x,y)$. According to E. L. Ince[4, Chapter 3], $\phi(x,y_0)$ has continuous first-order partial derivatives on $\overline{\Omega}$. Moreover, it is not hard to prove that the function $\phi(x,y_0)$ is twice continuously differentiable with respect to y_0 on $\overline{\Omega}$ by following [4, p.69]. Thus, from equation (2.2), it follows that ϕ has continuous second-order partial derivatives on $\overline{\Omega}$.

Lemma 2.2 *The function* $\psi(x,y)$ *has continuous second-order partial derivatives on* $\overline{\Omega}$*.*

Proof: First we show that $\psi(x,y)$ is twice continously differentiable with respect to x on $\overline{\Omega}$. For each point (x,y) in $\overline{\Omega}$. By the definition of ψ, one has

$$\begin{aligned}&y = \phi(x,\psi(x,y)),\\&y+k = \phi(x+h,\psi(x+h,y+k)).\end{aligned}$$

Hence

$$\begin{aligned}&\phi(x+h,\psi(x+h,y+k)) - \phi(x,\psi(x,y))\\&= \phi(x+h,\psi(x+h,y+k)) - \phi(x+h,\psi(x,y)) + \phi(x+h,\psi(x,y)) - \phi(x,\psi(x,y))\\&= \phi_{y_0}(x+h,\bar{y}_0)[\psi(x+h,y+k) - \psi(x,y)] + [\phi(x+h,\psi(x,y)) - \phi(x,\psi(x,y))]\\&= k.\end{aligned} \tag{2.3}$$

Here $\bar{y}_0$ comes from mean value theorem. So

$$\begin{aligned}&|\phi_{y_0}(x+h,\bar{y}_0)[\psi(x+h,y+k) - \psi(x,y)]|\\&\le |k| + |\phi(x+h,\psi(x,y)) - \phi(x,\psi(x,y))|\\&\le |k| + M|h|,\end{aligned}$$

where M is positive constant such that $|\phi_x| \le M$ on $\overline{\Omega}$. From the fact that $\phi_{y_0} > 0$ on $\overline{\Omega}$, it follows that ψ is continuous at (x,y). Furthermore, dividing both sides of the last equality of the sequence of formulas (2.3) for $k = 0$ by h, one has

$$\frac{\psi(x+h,y) - \psi(x,y)}{h} = -\frac{\phi(x+h,\psi(x,y)) - \phi(x,\psi(x,y))}{h} \times \frac{1}{\phi_{y_0}(x+h,\bar{y}_0)}.$$

As h tends to zero, the right side of above equation tends to $-\frac{\phi_x(x,\psi(x,y))}{\phi_{y_0}(x,\psi(x,y))}$. Thus the derivative of ψ with respect to x at the point (x,y) exists and

$$\psi_x(x,y) = -\frac{\phi_x(x,\psi(x,y))}{\phi_{y_0}(x,\psi(x,y))}. \tag{2.4}$$

Since the point (x,y) is arbitrary in $\overline{\Omega}$, $\psi(x,y)$ is continuously differentiable with respect to x on $\overline{\Omega}$. From formula (2.4), it is clearly that the derivative $\psi_x(x,y)$ is continuously differentiable with respect to x on $\overline{\Omega}$. Now, fix $x \in [0,a]$. $\psi(x,\cdot)$ is the inverse of $\phi(x,\cdot)$. Since ϕ is twice continuously differentiable with respect to y_0 on $\overline{\Omega}$, it is clearly that ψ is twice continuously differentiable with respect to y on $\overline{\Omega}$. Finally, from formula (2.4), one obtains that the derivative $\psi_x(x,y)$ is continuously differentiable with respect to y on $\overline{\Omega}$. □

The function v_0 is now easily determined, and the result is

$$v_0(x,y) = \int_0^x \exp\left\{-\int_s^x \frac{q(\tau,\phi(\tau,\psi(x,y)))}{p(\tau,\phi(\tau,\psi(x,y)))}\,d\tau\right\} \frac{f(s,\phi(s,\psi(x,y)))}{p(s,\phi(s,\psi(x,y)))}\,ds.$$

Then we have

$$L(u - v_0) = \varepsilon\Delta v_0.$$

Moreover, it follows that Δv_0 is uniformly bounded on $\overline{\Omega}$ from preceding argument regarding functions ϕ and ψ, and the assumptions for p, g, q and f. It is quite evident that this approximation for $u(x,y,\varepsilon)$ is not valid in a neighborhood of three parts of the boundary of Ω, $x = a$, $y = 0$, and $y = b$, and is valid in the remaining subregion of Ω including the neighborhood of the boundary $x = 0$ up to the order $O(\varepsilon)$. To obtain a "uniform" approximation of the solution u in Ω, we eliminate these discrepancies along the boundaries $x = a$, $y = 0$, and $y = b$ by introducing other functions, called boundary layer functions, along the three boundaries.

3 The Boundary Layer Functions

In Chapter 2, we obtained an approximate solution v_0, to (1.1) by solving the reduced differential equation. Unfortunately, this approximate solution does not satisfy the given boundary conditions. The boundary layer functions, which we now define, are designed to correct this discrepancy in the boundary data of $u - v_0$. For this, we introduce a stretched variable $\eta = y/\sqrt{\varepsilon}$, and make a formal expansion in powers of $\sqrt{\varepsilon}$. we note the formula

$$Lw(x,\eta) = -\varepsilon w_{xx}(x,\eta) - w_{\eta\eta}(x,\eta) + p(x,y)w_x(x,\eta) + g(x,y)\frac{1}{\sqrt{\varepsilon}}w_\eta(x,\eta) + q(x,y)w(x,\eta).$$

We make a formal expansion of the equation $Lw = 0$ into powers of $\sqrt{\varepsilon}$, and equate to zero the coefficients of powers of $\varepsilon^{\frac{1}{2}}$, to define $w_0^{(1)}$ by

$$-w_{0\eta\eta}^{(1)} + p(x,0)w_{0x}^{(1)} + g_y(x,0)\eta w_{0\eta}^{(1)} + q(x,0)w_0^{(1)} = 0, \tag{3.1}$$

We must specify boundary conditions to complete the determination of the $w_0^{(1)}$. We specify the boundary conditions

$$w_0^{(1)}(x,0) = -v_0(x,0), \tag{3.1a}$$

$$w_0^{(1)}(0,\eta) = 0, \tag{3.1b}$$

$$w_0^{(1)}(x,\infty) = 0. \tag{3.1c}$$

In order to obtain a solution, we make the following change of independent variables:

$$t = \int_0^{h(x)} e^{-2\int_0^{h^{-1}(\tau)} \frac{g_y(s,0)}{p(s,0)}ds} d\tau, \quad \xi = \eta e^{-\int_0^x \frac{g_y(s,0)}{p(s,0)}ds},$$

and introduce the new dependent variable

$$w(t,\xi) = e^{k(x)} w_0^{(1)}(x,\eta),$$

where

$$h(x) = \int_0^x \frac{1}{p(s,0)} ds, \qquad k(x) = \int_0^x \frac{q(s,0)}{p(s,0)} ds$$

for all $x \in [0,a]$. Here h^{-1} is the inverse of h. A computation then gives

$$-w_{\xi\xi} + w_t = 0 \tag{3.2}$$

with the boundary conditions

$$\begin{aligned} &w(t,0) = -e^{k(x)} v_0(x,0), \\ &w(0,\xi) = 0, \\ &w(t,\infty) = 0. \end{aligned}$$

We then have

$$w(t,\xi) = -\sqrt{\frac{2}{\pi}} \int_{\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-\frac{s^2}{2}} \int_0^{r^{-1}(t-\frac{\xi^2}{2s^2})} e^{\int_0^\tau \frac{q(\zeta,0)}{p(\zeta,0)} d\zeta} \frac{f(\tau,0)}{p(\tau,0)} d\tau ds, \tag{3.3}$$

where $r(x) = \int_0^{h(x)} e^{-2\int_0^{h^{-1}(\tau)} \frac{g_y(s,0)}{p(s,0)} ds} d\tau$ for all $x \in [0,a]$ and r^{-1} is the inverse of r. It follows that the problem (3.1), $(3.1a,b,c)$ has a unique solution $w_0^{(1)}$.

Lemma 3.1 *Under the hypotheses* $(C1)-(C4)$, *One obtains*

$$\begin{aligned} &|D_t^i w| \le C e^{-\frac{\xi}{\sqrt{2t}}} \quad \textit{for} \quad i = 0,1, \\ &|D_\xi w| \le C \tfrac{t}{\xi} e^{-\frac{\xi}{\sqrt{2t}}}, \\ &|D_t D_\xi w| \le C[\tfrac{1}{\sqrt{2t}} e^{-\frac{\xi}{\sqrt{2t}}} + \tfrac{t}{\xi} e^{-\frac{\xi}{\sqrt{2t}}}], \\ &|D_\xi^2 w| \le C e^{-\frac{\xi}{\sqrt{2t}}}, \end{aligned}$$

where C is a positive constant independent of ε.

Proof: It is easy to see from formula (3.3) that

$$|D_t^i w| \le C \int_{\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-\frac{s^2}{2}} ds$$

for $i = 0,1$,

$$|D_\xi w| \le C \int_{\frac{\xi}{\sqrt{2t}}}^{\infty} \frac{\xi}{s^2} e^{-\frac{s^2}{2}} ds,$$

and

$$|D_t D_\xi w| \le C[\frac{1}{\sqrt{2t}} e^{-\frac{\xi}{\sqrt{2t}}} + \int_{\frac{\xi}{\sqrt{2t}}}^{\infty} \frac{\xi}{s^2} e^{-\frac{s^2}{2}} ds],$$

where C is a positive constant independent of ε. Moreover,

$$e^{-\frac{s^2}{2}} \leq e^{\frac{1}{2}-s} = e^{\frac{1}{2}}e^{-s}$$

so

$$|D_t^i w| \leq C\int_{\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-s}ds = Ce^{-\frac{\xi}{\sqrt{2t}}}$$

for $i = 0,1,$

$$|D_\xi w| \leq C\frac{t}{\xi}\int_{\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-s}ds = C\frac{t}{\xi}e^{-\frac{\xi}{\sqrt{2t}}},$$

and

$$|D_t D_\xi w| \leq C[\frac{1}{\sqrt{2t}}e^{-\frac{\xi}{\sqrt{2t}}} + \frac{t}{\xi}\int_{\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-s}ds] = C[\frac{1}{\sqrt{2t}}e^{-\frac{\xi}{\sqrt{2t}}} + \frac{t}{\xi}e^{-\frac{\xi}{\sqrt{2t}}}],$$

where C is a positive constant independent of ε. From equation (3.2), we have

$$|D_\xi^2 w| = |D_t w| \leq Ce^{-\frac{\xi}{\sqrt{2t}}}$$

□

From above lemma, it follows that

$$|w_0^{(1)}(x,\eta)| = |e^{-k(x)}w(t,\xi)| \leq Ce^{-k(x)}e^{-\frac{\xi}{\sqrt{2t}}} \leq Ce^{-k(x)}e^{-\frac{\eta}{\sqrt{2h(x)}}}$$

Theorem 3.2 *Under the hypotheses* $(C1)-(C4)$, *one obtains*

$$|\eta w_0^{(1)}(x,\eta)| \leq Ce^{-k(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}},$$
$$|\eta D_x w_0^{(1)}(x,\eta)| \leq Ce^{-k(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}},$$
$$|\eta^2 D_\eta w_0^{(1)}(x,\eta)| \leq Ce^{-k(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}},$$

where C *is a positive constant independent of* ε.

Proof: From the formula

$$w_0^{(1)}(x,\eta) = e^{-k(x)}w(t,\xi)$$

and Lemma 3.1, we have

$$
\begin{aligned}
|\eta w_0^{(1)}(x,\eta)| &\leq C\eta e^{-k(x)}e^{-\frac{\eta}{\sqrt{2h(x)}}}\\
&\leq \frac{\eta}{\sqrt{2h(x)}}\sqrt{2h(x)}e^{-k(x)}e^{-\frac{\eta}{\sqrt{2h(x)}}}\\
&\leq Ce^{-k(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}},\\
|\eta D_x w_0^{(1)}(x,\eta)| &\leq Ce^{-k(x)}\left[|\eta w(t,\xi)|+e^{-2\int_0^x\frac{g_y(s,0)}{p(s,0)}ds}|\eta D_t w(t,\xi)|+|\eta\xi w_\xi(t,\xi)|\right]\\
&\leq Ce^{-k(x)}\left[\eta e^{-\frac{\xi}{\sqrt{2t}}}+\eta e^{-2\int_0^x\frac{g_y(s,0)}{p(s,0)}ds}e^{-\frac{\xi}{\sqrt{2t}}}+\eta t e^{-\frac{\xi}{\sqrt{2t}}}\right]\\
&\leq Ce^{-k(x)}\left[\eta e^{-\frac{\eta}{\sqrt{2h(x)}}}+\eta e^{-2\int_0^x\frac{g_y(s,0)}{p(s,0)}ds}e^{-\frac{\eta}{\sqrt{2h(x)}}}+\eta t e^{-\frac{\eta}{\sqrt{2h(x)}}}\right]\\
&\leq Ce^{-k(x)}\left[\sqrt{2h(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}}+\sqrt{2h(x)}e^{-2\int_0^x\frac{g_y(s,0)}{p(s,0)}ds}e^{-\frac{\eta}{2\sqrt{2h(x)}}}\right.\\
&\quad\left.+\sqrt{2h(x)}te^{-\frac{\eta}{2\sqrt{2h(x)}}}\right]\\
&\leq Ce^{-k(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}},\\
|\eta^2 D_\eta w_0^{(1)}(x,\eta)| &\leq e^{-k(x)}|\eta^2 D_\xi w(t,\xi)|e^{-\int_0^x\frac{g_y(s,0)}{p(s,0)}ds}\\
&\leq Ce^{-k(x)}\eta te^{-\frac{\eta}{\sqrt{2h(x)}}}\\
&\leq Ce^{-k(x)}\sqrt{2h(x)}te^{-\frac{\eta}{2\sqrt{2h(x)}}}\\
&\leq Ce^{-k(x)}e^{-\frac{\eta}{2\sqrt{2h(x)}}},
\end{aligned}
$$

where C is a positive constant independent of ε. □

By above analysis, the weighted derivatives $\eta D_x w_0^{(1)}$ and $\eta^2 D_\eta w_0^{(1)}$ are uniformly bounded in $\overline{\Omega}$. However,

$$
D_x^2 w_0^{(1)} = \sqrt{\frac{2}{\pi}}\left\{-\frac{\xi}{(2t)^{\frac{3}{2}}}e^{-\frac{\xi^2}{4t}}f(0,0)e^{-4\int_0^x\frac{g_y(s,0)}{p(s,0)}ds}h'(x)^2+I\right\}e^{-k(x)},
$$

where I is a function that is uniformly bounded in $\overline{\Omega}$. The derivative thus posses singularity at the origin.

In a manner similar to the construction of the $w_0^{(1)}$, we define a stretched variable $\overline{\eta}=\frac{b-y}{\sqrt{\varepsilon}}$, and we define the function $w_0^{(2)}$ satisyind the equation

$$
-w^{(2)}_{0\overline{\eta}\overline{\eta}}+p(x,b)w^{(2)}_{0x}+g_y(x,b)\overline{\eta}w^{(2)}_{0\overline{\eta}}+q(x,b)w_0^{(2)}=0,
$$

with he boundary conditions

$$
\begin{aligned}
&w_0^{(2)}(x,0)=-v_0(x,b),\\
&w_0^{(2)}(0,\overline{\eta})=0,\\
&w_0^{(2)}(x,\infty)=0.
\end{aligned}
$$

Analogously, the weighted derivatives $\bar{\eta} D_x w_0^{(2)}$ and $\bar{\eta}^2 D_\eta w_0^{(2)}$ are uniformly bounded in $\overline{\Omega}$. However, the derivative $D_x^2 w_0^{(2)}$ posses singularity at the point $x = 0$, $y = b$. We also define the functions $R_0^{(1,2)}$ by the relation

$$R_0^{(1,2)} = L[w_0^{(1,2)}].$$

Then

$$\begin{aligned} R_0^{(1)} &= -\varepsilon w_{0xx}^{(1)} \\ &+ \sqrt{\varepsilon}\left\{ \tfrac{p(x,y)-p(x,0)}{\sqrt{\varepsilon}} w_{0x}^{(1)} + \tfrac{g(x,y)-g_y(x,0)y}{\varepsilon} w_{0\eta}^{(1)} + \tfrac{q(x,y)-q(x,0)}{\sqrt{\varepsilon}} w_0^{(1)} \right\} \end{aligned}$$

and

$$\begin{aligned} R_0^{(2)} &= -\varepsilon w_{0xx}^{(2)} \\ &+ \sqrt{\varepsilon}\left\{ \tfrac{p(x,y)-p(x,b)}{\sqrt{\varepsilon}} w_{0x}^{(2)} - \tfrac{g(x,y)-g_y(x,b)(y-b)}{\varepsilon} w_{0\eta}^{(2)} + \tfrac{q(x,y)-q(x,b)}{\sqrt{\varepsilon}} w_0^{(2)} \right\}. \end{aligned}$$

According to the hypotheses $(C2)$, $(C3)$, and $(C4)$, we obtain the estimate

$$R_0^{(1,2)} = O(\sqrt{\varepsilon})$$

valid uniformly in $\overline{\Omega}$, with the exception of small fixed neighborhood of the corner points $x = 0$, $y = 0$ and $x = 0$, $y = b$. The estimate is valid in, say, $\delta \le x \le a$, $0 \le y \le b$, where δ is a small fixed positive number. However, we shall need estimate valid in a more extended region G, defined by $\varepsilon^\alpha \le x \le a$, $0 \le y \le b$, where $0 < \alpha < 1$. The estimate follows from analysis of corner singularities that are present in $R_0^{(1,2)}$. Using the inequality

$$\frac{\xi}{(4t)^{\frac{1}{2}}} e^{-\frac{\xi^2}{4t}} < 1,$$

valid uniformly in $0 \le x \le a$, $0 \le y \le b$, we find the estimate

$$R_0^{(1,2)} = O(\varepsilon^{min(\frac{1}{2}, 1-\alpha)})$$

in G. We write

$$u = v_0 + w_0^{(1)} + w_0^{(2)} + Z.$$

There remains then to be solved the following problem:

$$\begin{aligned} &-\varepsilon \Delta Z + pZ_x + gZ_y + qZ = \varepsilon \Delta v_0 - R_0^{(1)} - R_0^{(2)}, \\ &Z(0,y) = 0, \\ &Z(a,y) = -v_0(a,y) - w_0^{(1)}(a, \tfrac{y}{\sqrt{\varepsilon}}) - w_0^{(2)}(a, \tfrac{b-y}{\sqrt{\varepsilon}}), \\ &Z(x,0) = -w_0^{(2)}(x, \tfrac{b}{\sqrt{\varepsilon}}), \\ &Z(x,b) = -w_0^{(1)}(x, \tfrac{b}{\sqrt{\varepsilon}}). \end{aligned}$$

We find that Z is not very small near the boundary $x = a$. For this, we introduce a stretched variable $X = \frac{a-x}{\varepsilon}$. Note that

$$LZ(X,y) = -\frac{1}{\varepsilon}Z_{XX} - \varepsilon Z_{yy} - \frac{1}{\varepsilon}pZ_X + gZ_y + qZ.$$

We make a formal expansion of the equation $LZ = 0$ into powers of ε, and equate to zero the coefficients of powers of ε, to define Z_0, Z_1, Z_2 by

$$\begin{aligned} Z_{0XX} + p(a,y)Z_{0X} &= 0, \\ Z_{1XX} + p(a,y)Z_{1X} &= p_x(a,y)XZ_{0X} + g(a,y)Z_{0y} + q(a,y)Z_0, \\ Z_{2XX} + p(a,y)Z_{2X} &= -Z_{0yy} - \tfrac{1}{2}p_{xx}(a,y)X^2Z_{0X} + p_x(a,y)XZ_{1X} - g_x(a,y)XZ_{0y} \\ &\quad + g(a,y)Z_{1y} + q(a,y)Z_1 - g_x(a,y)XZ_0. \end{aligned}$$

We use the boundary conditions

$$\begin{aligned} &Z_0(0,y) = \psi(y), \;\; Z_1(0,y) = 0, \;\; Z_2(0,y) = 0, \\ &Z_i(X,y) \to 0 \;\; \text{as} \;\; X \to \infty, \;\; i = 0,1,2, \end{aligned}$$

where $\psi(y) = -v_0(a,y) - w_0^{(1)}(a, \frac{y}{\sqrt{\varepsilon}}) - w_0^{(2)}(a, \frac{b-y}{\sqrt{\varepsilon}})$. For the function

$$Z^*(X,y,\varepsilon) = Z_0(X,y,\varepsilon) + \varepsilon Z_1(X,y,\varepsilon) + \varepsilon^2 Z_2(X,y,\varepsilon),$$

we then have

$$L(Z^*) = R_2;$$

R_2 is given by

$$\begin{aligned} R_2 \;=\; & -\varepsilon^2 Z_{1yy} - \varepsilon^3 Z_{2yy} + \tfrac{1}{6}\varepsilon^2 p_{xxx}(a_1,y)X^3Z_{0X} - \tfrac{1}{2}\varepsilon^2 p_{xx}(a_2,y)X^2Z_{1X} \\ & + \varepsilon^2 p_x(a_3,y)XZ_{2X} + \tfrac{1}{2}\varepsilon^2 g_{xx}(a_4,y)X^2Z_{0y} - \varepsilon^2 g_x(a_5,y)XZ_{1y} + \varepsilon^2 g(x,y)Z_{2y} \\ & + \varepsilon^2 q(x,y)Z_2 + \tfrac{1}{2}\varepsilon^2 q_{xx}(a_6,y)X^2Z_0 - \varepsilon^2 q_x(a_7,y)XZ_1. \end{aligned}$$

Here $a_1, a_2, a_3, etc.$ come from Taylor Theorem. The explicit calculation of the functions Z_1 and Z_2 is complicated, but these functions can be shown to be of the following general form:

$$\begin{aligned} Z_0(X,y,\varepsilon) &= \psi(y)e^{-p(a,y)X}, \\ Z_1(X,y,\varepsilon) &= \left\{[r_1(y)X^2 + r_2(y)X]\psi(y) + r_3(y)X\tfrac{d}{dy}\psi\right\}e^{-p(a,y)X}, \\ Z_2(X,y,\varepsilon) &= \Big\{[(s_1(y)X^4 + s_2(y)X^3 + s_3(y)X^2 + s_4(y)X]\psi(y) \\ &\quad + [s_5(y)X^3 + s_6(y)X^2 + s_7(y)X]\tfrac{d}{dy}\psi + s_8(y)X\tfrac{d^2}{dy^2}\psi\Big\}e^{-p(a,y)X}, \end{aligned}$$

where the functions $s_1, s_2, r_1, etc.$ are all uniformly bounded and as are their derivatives with respect to y. Now we have

$$\frac{d^n}{dy^n}\psi = O(\varepsilon^{-\frac{1}{2}n}) \;\; \text{for} \;\; n = 0,1,2,3,4.$$

It follows that

$$R_2 = O(\sqrt{\varepsilon}).$$

Returning to the boundary value problem $(1.1),(1.1a)$, we summarize our results by writing

$$u(x,y,\varepsilon) = v_0(x,y) + w_0^{(1)}(x, \frac{y}{\sqrt{\varepsilon}}) + w_0^{(2)}(x, \frac{b-y}{\sqrt{\varepsilon}}) + Z^*(\frac{a-x}{\varepsilon}, y, \varepsilon) + \phi(x,y,\varepsilon). \tag{3.4}$$

The remainder term ϕ is a solution of the equation

$$L(\phi) = R;$$

R is given by

$$R = -\varepsilon\Delta v_0 - R_0^{(1)} - R_0^{(2)} - R_2.$$

In the subregion G, we have

$$R = O(\varepsilon^{\min(\frac{1}{2},1-\alpha)}).$$

We proceed to analyse the boundary conditions for the function ϕ. These are

$$\begin{aligned}
&\phi(0,y) = -Z^*(\tfrac{a}{\varepsilon}, y, \varepsilon),\\
&\phi(a,y) = 0,\\
&\phi(x,0) = -w_0^{(2)}(x, \tfrac{b}{\sqrt{\varepsilon}}) - Z^*(\tfrac{a-x}{\varepsilon}, 0, \varepsilon),\\
&\phi(x,b) = -w_0^{(1)}(x, \tfrac{b}{\sqrt{\varepsilon}}) - Z^*(\tfrac{a-x}{\varepsilon}, b, \varepsilon).
\end{aligned}$$

Utilizing the explicit results for the boundary layer functions, we have

$$\phi = O(\sqrt{\varepsilon}) \quad \text{on} \quad \Gamma. \tag{3.5}$$

4 Asymtotic Representation of the Solution

In order to estimate the remainder term $\phi(x,y,\varepsilon)$ in (3.4), we need the following result, which is a consequence of the maximum principle.

Theorem 4.1 *If $\Phi(x,y,\varepsilon)$ is the solution of the boundary value problem*

$$L[\Phi] = h(x,y,\varepsilon),$$

valid in Ω with

$$\Phi(x,y,\varepsilon)|_\Gamma = \Psi(x,y,\varepsilon)|_\Gamma$$

along the boundary Γ *of* Ω*, and if*

$$h(x,y,\varepsilon) = O(\varepsilon^{\mu}) \ \ in \ \ \overline{\Omega}, \ \mu \geq 0,$$

and

$$\Psi(x,y,\varepsilon) = O(\varepsilon^{\nu}) \ \ along \ \ \Gamma, \ \nu \geq 0,$$

then at most

$$\Phi(x,y,\varepsilon) = O\left(\varepsilon^{\min(\mu,\nu)}\right) \ \ in \ \ \overline{\Omega}.$$

Let us define the subregion G^*:

$$(x,y) \in G^* \ \text{ if } \ 0 \leq x \leq \varepsilon^{\alpha}, \ 0 \leq y \leq b, \ \ 0 < \alpha < 1.$$

It follows from Theorem 2.1 that

$$u(x,y,\varepsilon) = O(\varepsilon^{\alpha}) \ \text{ in } \ G^*.$$

Moreover, using in equation (3.4) the explicit definition of the function v_0, $w_0^{(1,2)}$, and Z^*, one finds without difficult that

$$\phi = O(\varepsilon^{\alpha}) \ \text{ in } \ G^*. \tag{4.1}$$

We next consider the subregion G. In this region we have the equation

$$L(\phi) = R = O(\varepsilon^{\min(\frac{1}{2},1-\alpha)}). \tag{4.2}$$

Moreover, result (4.1) also holds at the boundary $x = \varepsilon^{\alpha}$ of G. Together with the result (3.5) we obtain

$$\phi = O(\varepsilon^{\min(\frac{1}{2},\alpha)}) \ \ \text{ on } \ \ \partial G. \tag{4.3}$$

To the boundary value problem $(4.2), (4.3)$, we apply Theorem 4.1, it follows that

$$\phi = O(\varepsilon^{\min(\frac{1}{2},\alpha,1-\alpha)}) \ \text{ in } \ G. \tag{4.4}$$

Combining results (4.1) and (4.4), we obtain

$$\phi = O(\varepsilon^{\min(\frac{1}{2},\alpha,1-\alpha)}) \ \text{ in } \ \overline{\Omega}.$$

Hence the optimal choice of α is $\alpha = \frac{1}{2}$ and then

$$\phi = O(\sqrt{\varepsilon}) \ \text{ in } \ \overline{\Omega}.$$

Finally we analyze more closely the boundary layer function Z appearing in that expansion. From the explicit formulas one easily finds

$$Z^* = \psi(y)\exp\left[-p(a,y)\frac{a-x}{\varepsilon}\right] + O(\sqrt{\varepsilon}).$$

Hence we have established the following theorem:

Theorem 4.2 *If the function u satisfies the boundary value problem* (1.1), $(1.1a)$, *then uniformly in* $\overline{\Omega}$, *including all four corner points,*

$$u(x,y,\varepsilon) = v_0(x,y) + w_0^{(1)}(x, \frac{y}{\sqrt{\varepsilon}}) + w_0^{(2)}(x, \frac{b-y}{\sqrt{\varepsilon}}) + \psi(y)\exp\left[-p(a,y)\frac{a-x}{\varepsilon}\right] + O(\sqrt{\varepsilon}).$$

References

[1] Eckhaus, W., and de Jager, E.M. (1966) Asymptotic Solutions of Singular Perturbation Problems for Linear Differential Equations of Elliptic Type. *Arch. Rational Mech. Anals.* **23**, 26–86.

[2] Grisvard, P. (1985) *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston.

[3] Il'in, A.M. (1992) *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, American Mathematical Society, Rhode Island.

[4] Ince, E.L. (1956) *Ordinary Differential Equations*, Dover Pubications, New York.

[5] Kellogg, R.B. and Shih, S.D. (1987) Asymptotic Analysis of a Singular Perturbation Problem. *SIAM J. Math. Anal.* **18**, 1467–1511.

[6] Levinson, N. (1950) The First Boundary Value Problem for $\varepsilon\Delta u + Au_x + Bu_y + Cu = D$ for Small ε. *Annals of Math.* **51**, 428–455.

[7] Vasil'eva, A.B., Butuzov, V.F., and Kalachev, L.V. (1993) *The Boundary Function Method for Singular Perturbation Problems*, Springer Verlag, New York.

一個奇界攝動問題解的漸近形式

盧性良*

摘　要

我們探討橢圓偏微方程

$$Lu \equiv -\varepsilon\,\Delta u + pu_x + gu_y + qu = f\,,$$

在四方形 $\Omega = (0,a)\times(0,b)$ 上的 Dirichlet 邊界值問題，其中 $0<\varepsilon<<1$，Δ是 Laplacian 運算，而且 p, g, q 和 f 滿足某些假設；特別是 $p>0$，$q \geq 0$。對很小的 ε，我們構造這個問題的解的一個形式的漸近展開。這個展開式包含退化方程的解和邊界層函數。拋物線邊界層函數滿足一個拋物線方程，其具有一個無界的係數。我們轉換此拋物線方程成一個熱方程，以此來討論拋物線邊界層函數的性質。最後這個展開式的剩餘項被估計為 ε 的次方。

關鍵詞：奇界攝動，邊界層。

* 東海大學數學系